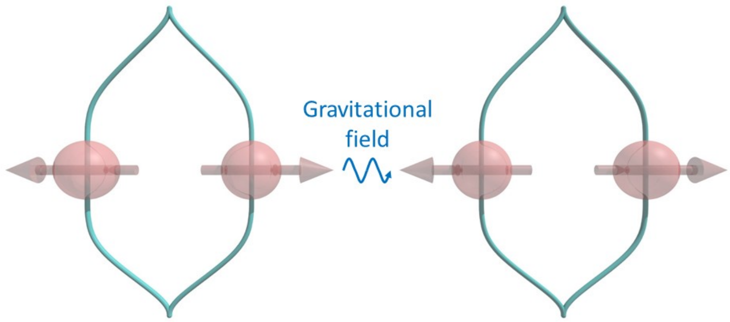
Tutorials – Macroscopic superpositions to test quantum gravity
International school on Quantum Sensors for Fundamental Physics QSFP 2021

Welcome to the tutorials!



What is the gravitational field generated by a quantum system?

The material is split into 4 exercises:

1. Basic interferometry and gravitational phases
2. Gravitational field of a Superposition
3. Decoherence in matter-wave interferometers with nanoparticles
4. State-of-the-art-experiments and future prospects

Parts (a),(b) of each exercise contains the core material, while part (c) is given as a bonus exercise.

If you have any questions feel free to contact us at marko.toros(at)glasgow.ac.uk, julen.pedernales(at)gmail.com, a.pontin(at)ucl.ac.uk, giulio.gasbarri(at)uab.ac.

1) BASIC INTERFEROMETRY AND GRAVITATIONAL PHASES

We consider a non-relativistic mass m with initial kinetic energy $E_0 = p_0^2/2m$ in three different situations depicted in Fig. 1. The key message of this exercise is that the interferometric phases are due to *potential differences*.

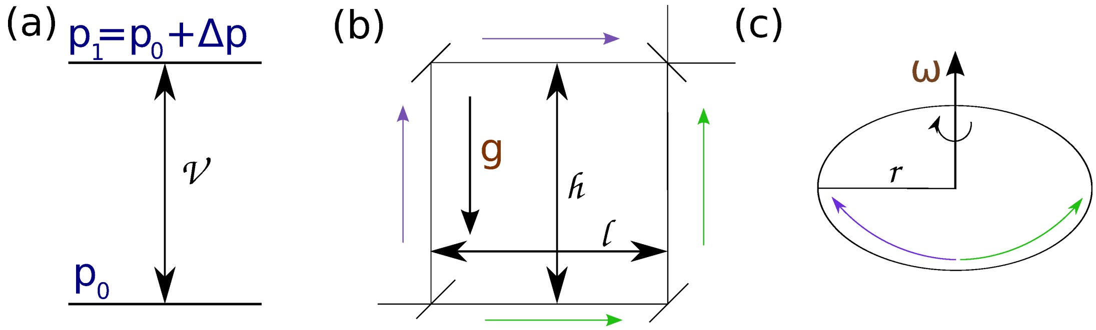


Figure 1: (a) Potential difference V induces a momentum change $p_0 \to p_0 + \Delta p$. (b) Interferometer in the Earth's gravitational field (g is the gravitational acceleration). (c) Interferometer on a rotating platform rotating with angular frequency ω.

(a) The particle traverses a potential difference V. Starting from energy conservation

$$\frac{p_0^2}{2m} = \frac{p_1^2}{2m} + V, \tag{1}$$

and assuming V is such that the corresponding momentum change is small (i.e. $p_1 - p_0 = \Delta p \ll p_0$), compute the relative change in momentum

$$\frac{\Delta p}{p_0} = -\frac{1}{2}\frac{V}{E_0}. \tag{2}$$

(b) The phase difference $\Delta\phi$ between the paths indicated by purple/green arrows can be computed as a

$$\Delta\phi = \frac{1}{\hbar}\oint \boldsymbol{p}\cdot d\boldsymbol{r} = \frac{\Delta p l}{\hbar}, \tag{3}$$

where l is the length of the horizontal segment (the phase difference arises from the gravitational *potential* difference $V = mgh$). Using Eq. (2) compute the Collella-Overhauser-Werner phase difference:

$$\Delta\phi_{\mathrm{COW}} = -\frac{m^2 g\lambda}{2\pi\hbar^2} \times \mathcal{A}, \tag{4}$$

where the area of the interferometer is $\mathcal{A} = lh$, and the matterwave wavelength is $\lambda = 2\pi\hbar/(p_0)$.

(c) The phase difference $\Delta\phi$ between the paths indicated by purple/green arrows can be again computed using Eq. (3). The potential difference is given by $V = \boldsymbol{\omega}\cdot\boldsymbol{L}$, where $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{p}_0$ is the angular momentum. Using Eqs. (2) and (3) compute the Sagnac phase difference:

$$\Delta\phi_{\mathrm{Sagnac}} = \frac{2m\omega}{\hbar} \times \mathcal{A}, \tag{5}$$

where the area of the interferometer is $\mathcal{A} = \pi r^2$. Find the condition for $\Delta\phi_{\mathrm{COW}}/\Delta\phi_{\mathrm{Sagnac}} \sim 1$.

References:
[a] H. Rauch et al, Neutron Interferometry, Oxford University Press, USA (2015)
[b] R. Colella et al, Phys. Rev. Lett. 34, 1472 (1975)
[c] S. A. Werner et al, Phys. Rev. Lett. 42, 1103 (1979)

2) GRAVITATIONAL FIELD OF A SUPERPOSITION

We consider a non-relativistic particle s_1 with mass m_1 in a spatial superposition state:

$$|\phi_1\rangle = \frac{|\alpha\rangle + |\beta\rangle}{\sqrt{2}}, \tag{6}$$

where $\langle x_1|\alpha\rangle = \delta(x_1 - d)$ and $\langle x_1|\beta\rangle = \delta(x_1 + d)$. We suppose for simplicity that $|\phi_1\rangle$ will not evolve thus providing a constant source for the gravitational field. We further consider a test particle s_2 with mass m_2 in an initial gaussian state centered in the origin, i.e.

$$\langle x_2|\phi_2\rangle = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{x_2^2}{4\sigma^2}}. \tag{7}$$

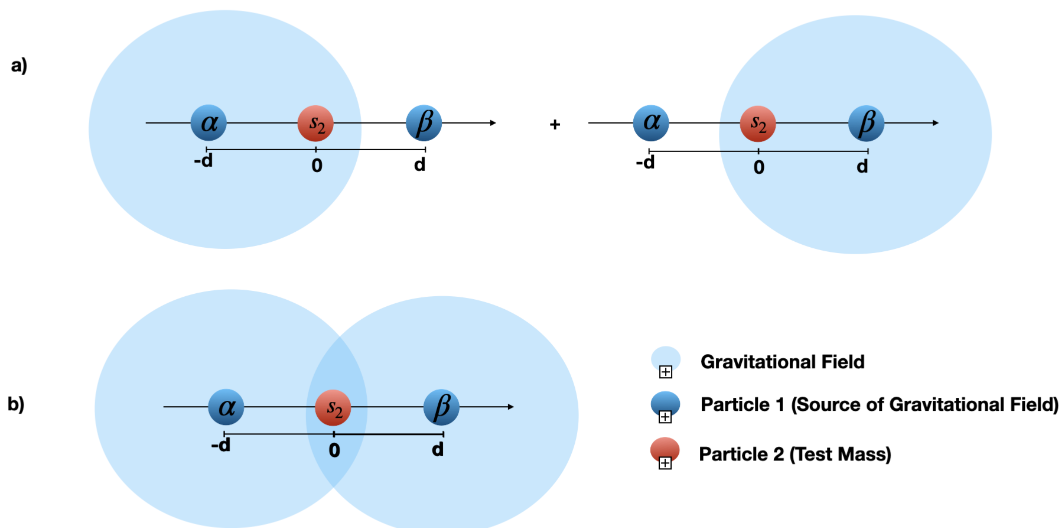


Figure 2: The particle s_1, which generates the gravitational field is in a superposition of the states α and β. Particle s_2, which is used as test mass, is in a localized state located in the center.
(a) Quantum Case: The total gravitational field is a linear combination of gravitational fields produced by the states α and β. (b) Semi-classical treatment of gravity: the total gravitational field acting on m_2 is produced by the mass density $m_1(|\alpha(x)|^2 + |\beta(x)|^2)/2$.

(a) In a quantum scenario we would expect the gravitational field to be in a superposition of the gravitational fields generated independently by $|\alpha\rangle$ and $|\beta\rangle$. Suppose the initial state is $|\phi_1\rangle|\phi_2\rangle$.

1) Convince yourself that the state after the evolution will take the following form:

$$|\phi_{qg}, t\rangle = \frac{1}{\sqrt{2}} \left(|\phi_\alpha, t\rangle|\alpha\rangle + |\phi_\beta, t\rangle|\beta\rangle \right), \tag{8}$$

where the states of the test particle satisfy the following dynamics:

$$\begin{aligned} i\hbar\partial_t|\phi_\alpha, t\rangle &= \left\{ \frac{\hat{p}_2^2}{2m_2} + \frac{Gm_1m_2}{|\hat{x}_2 - d|} \right\} |\phi_\alpha, t\rangle \\ i\hbar\partial_t|\phi_\beta, t\rangle &= \left\{ \frac{\hat{p}_2^2}{2m_2} + \frac{Gm_1m_2}{|\hat{x}_2 + d|} \right\} |\phi_\beta, t\rangle \end{aligned} \tag{9}$$

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Hint: the state $|\phi_\alpha\rangle$ feels the gravitational potential generated by $|\alpha\rangle$ while the state $|\phi_\beta\rangle$ feels the gravitational potential generated by $|\beta\rangle$.

2) Assuming that the gravitational potential can be approximated as

$$\frac{Gm_1m_2}{|\hat{x}_2 \pm d|} \simeq \frac{Gm_1m_2}{|d|} \mp \frac{Gm_1m_2}{|d|^2}\hat{x}_2, \tag{10}$$

it can be shown that

$$\langle x_2|\phi_\alpha, t\rangle = \frac{1}{\{2\pi\sigma^2[1+(\hbar t/m_2\sigma)^2)]\}^{1/4}} \exp\left(-\frac{(x_2 + \frac{Gm_1}{2|d|^2}t^2)^2}{4\sigma^2(1+(\frac{\hbar t}{m_2\sigma})^2)}\right) \tag{11}$$

$$\langle x_2|\phi_\beta, t\rangle = \frac{1}{\{2\pi\sigma^2[1+(\hbar t/m_2\sigma)^2)]\}^{1/4}} \exp\left(-\frac{(x_2 - \frac{Gm_1}{2|d|^2}t^2)^2}{4\sigma^2(1+(\frac{\hbar t}{m_2\sigma})^2)}\right). \tag{12}$$

Verify that the states $|\phi_\alpha, t\rangle$ and $|\phi_\beta, t\rangle$ are localized around the following positions

$$\langle\phi_\alpha, t|\hat{x}_2|\phi_\alpha, t\rangle = -\frac{Gm_1}{2|d|^2}t^2, \qquad \langle\phi_\beta, t|\hat{x}_2|\phi_\beta, t\rangle = \frac{Gm_1}{2|d|^2}t^2 \tag{13}$$

while the mean value of the total state is still zero, i.e.

$$\langle\phi_{qg}, t|\hat{x}_2|\phi_{qg}, t\rangle = \frac{1}{2}(\langle\phi_\alpha, t|\hat{x}_2|\phi_\alpha, t\rangle + \langle\phi_\beta, t|\hat{x}_2|\phi_\beta, t\rangle) = 0. \tag{14}$$

(b) In a semi-classical scenario the gravitational field is not allowed to be in a superposition of two different configurations. Let us suppose the gravitational field is sourced by the mass density $m_1|\phi_1(x)|^2$.
Assume the initial state of the test and source particle to be again $|\phi_1\rangle|\phi_2\rangle$.

1) Convince yourself that, after the evolution, the state will take the following form:

$$|\phi_{cl}, t\rangle = |\phi_{2,cl}, t\rangle\frac{|\alpha\rangle + |\beta\rangle}{\sqrt{2}} \tag{15}$$

and $|\phi_{2,cl}, t\rangle$ is governed by the following dynamical equation:

$$i\hbar\partial_t|\phi_{2,cl}, t\rangle = \left\{\frac{\hat{p}_2^2}{2m_2} + \hat{V}_{cl}(\hat{x}_2)\right\}|\phi_{2,cl}, t\rangle \tag{16}$$

where $V_{cl}(x) = \frac{1}{2}(\frac{Gm_1m_2}{|\hat{x}_2-d|} + \frac{Gm_1m_2}{|\hat{x}_2+d|})$.

2) Under the assumption in Eq. (10) show that:

$$V_{cl}(x) = \frac{Gm_1m_2}{|d|}. \tag{17}$$

Under this approximation can be shown that the state at time t reads

$$\langle x_2|\phi_{2,cl}, t\rangle = \frac{1}{\{2\pi\sigma^2[1+(\hbar t/m_2\sigma)^2)]\}^{1/4}} \exp\left(-\frac{x_2^2}{4\sigma^2(1+(\frac{\hbar t}{m_2\sigma})^2)}\right). \tag{18}$$

Show that:

$$\langle\phi_{cl}, t|\hat{x}_2|\phi_{cl}, t\rangle = 0. \tag{19}$$

(c) In the scenario described in Fig. 2, measuring position of the test particle is not enough to discriminate between a quantum and a semi-classical scenario but measuring variance of the test particle maybe help on testing the two hypothesis. Show that:

$$\langle\phi_{qg},t|\hat{x}_2^2|\phi_{qg},t\rangle = \sigma^2\left(1+\left(\frac{\hbar t}{m_2\sigma}\right)^2\right)+\left(\frac{Gm_1}{2|d|^2}t^2\right)^2 \tag{20}$$

$$\langle\phi_{cl},t|\hat{x}_2^2|\phi_{cl},t\rangle = \sigma^2\left(1+\left(\frac{\hbar t}{m_2\sigma}\right)^2\right). \tag{21}$$

References:

[d] R. Feynman, et al. Feynman lectures on gravitation. CRC Press, 2018
[e] L. Diosi, Phys. Lett. 105A (1984) 199-202
[f] R. Penrose Found. Phys. 44, 557-575 (2014)
[g] D.Carney et al Class. Quantum Grav. 36 034001(2019)

3) DECOHERENCE IN MATTER-WAVE INTERFEROMETERS WITH NANOPARTICLES

Consider the one dimensional interferometric loop in Fig. 3a. The center of mass of a particle of mass m is prepared in a pure quantum state described by a Gaussian wave packet of width x_0 and zero momentum. The wave packet is split in two components, which travel in opposite directions with constant momenta $-\vec{k}$ and $\vec{k}$ until they reach a separation distance d at time $\tau/2$. At this moment, the wave packets are reflected and their momenta are inverted such that they now travel towards each other. At time τ, when the centers of the wave packets overlap, we measure the position of the particle. If the process is coherent, an interference pattern will be observed in the probability distribution of the center of mass position, which is of the form $P(x) = \frac{1}{N} e^{-\frac{x^2}{2\sigma_\tau^2}} \cos^2(\frac{kx}{\hbar} + \phi)$, where $k = \left|\vec{k}\right|$, $\sigma_\tau^2 = x_0^2[1 + (\frac{\hbar\tau}{x_0^2 2m})^2]$, N is a normalization constant and ϕ is a relative phase picked up between the components of the superposition as they travel different arms of the interferometric loop.

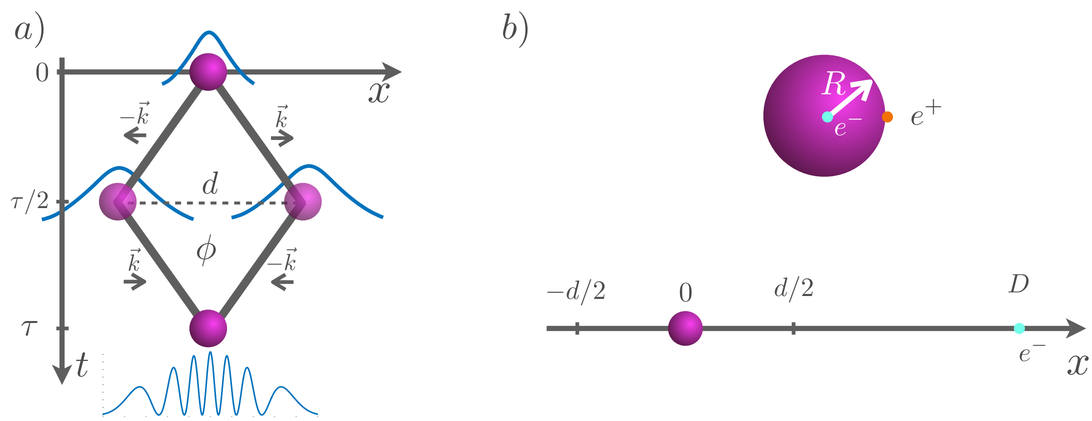

Figure 3: **Matter-wave interferometer**

a) To reconstruct the probability distribution, the experiment needs to be repeated a statistically significant number of times. Consider that the experiment is not initialized in a pure state of 0 momentum, but that it contains a finite temperature. One way to treat the problem is to consider that in each run of the experiment the initial wave packet will have a velocity v picked at random from a Boltzmann distribution according to the particles temperature. In one dimension, for a free particle of mass m such a distribution is given by $\mu_B(v) = \sqrt{\frac{m}{2\pi K_B T}} e^{-\frac{mv^2}{2K_B T}}$, where K_B is Boltzman's constant. Compute the effect of the initial temperature T of the center-of-mass motion on the visibility of the reconstructed interference pattern. [**Hint**: $\int dae^{-\frac{a^2}{2\sigma_a^2}} e^{-\frac{(x-a)^2}{\sigma^2}} \cos^2[k(x-a)] = Ae^{-\frac{x^2}{2(\sigma_a^2+\sigma^2)}}[1 + e^{-\frac{2k^2\sigma_a^2\sigma^2}{\sigma_a^2+\sigma^2}} \cos\left(\frac{2k\sigma^2}{\sigma_a^2+\sigma^2}x\right)]$, with A a constant]

b) Another source of noise that can erase the interference pattern is given by the interferometric phase ϕ. If the two arms of the interferometric loop observe different fields due to the presence of some uncontrolled environment, a relative phase will build up and this will shift the interference pattern. Shot-to-shot fluctuations of this phase will again reduce the visibility of the reconstructed interference pattern. Consider that the particle has an electric dipole moment given by the presence of a negative charge at the center of the particle and its compensating positive charge at the surface of the particle, as shown in Fig. 3b. Estimate the phase generated by a single electron sitting in the x axis at some distance D from the interferometer. For simplicity, you can assume that $D \gg d, R$ such that the field can be linearized across the size of the interferometer and the dipole can be assumed to observe a uniform electric field. You can also assume that the dipole is aligned with the electric field. [**Hint**: Remember the energy of an electric dipole moment $\vec{p}$ in an electric field $\vec{E}$ is given by $U = -\vec{p}\vec{E}$, and the phase picked by each component of the superposition can be computed as the time integral of its energy $\varphi = \int_0^\tau dt U(t)/\hbar$.]

c) Consider that the particle has a radius of $R = 250$ nm and that we are able to prepare it in a superposition of distance $d = R$, in a time $\tau = 0.5$ s. What is the minimal volume around the interferometer that we need to ensure is free of any electrical charges? [**Hint**: In S.I. units $e^- \approx -1.6 \cdot 10^{-19}$ C, $\hbar \approx 1.05 \cdot 10^{-34}$ J s, $\frac{1}{4\pi\epsilon_0} \approx 9 \cdot 10^9$ $\frac{\text{kgm}^3}{\text{s}^2\text{C}^2}$]

4) STATE-OF-THE-ART-EXPERIMENTS

This section will focus on two experimental results achieved in the past year. The first is the cooling to the motional ground state of a mesoscopic silica nanoparticle in an optical tweezer [h]. The second is a direct measurement of the gravitational attraction between two mg scale gold spheres [i]. The objective here is to perform order of magnitude estimations to get a feeling of the typical difficulties in these experiments.

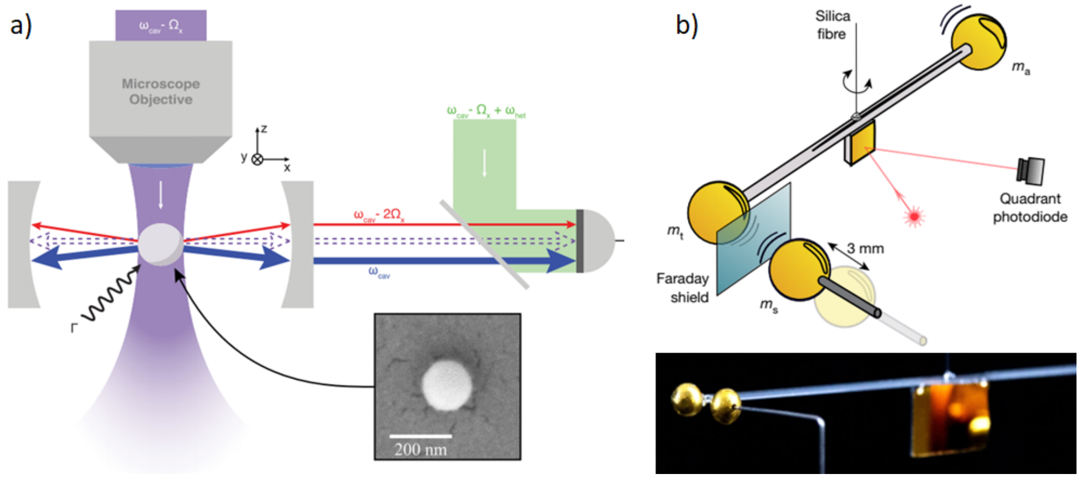

Figure 4: (a) Schematic overview of the ground state cooling experiment (taken from Ref. [h]). An optical tweezer provides a 3D trapping potential for a spherical silica nanoparticle. The motion along the x-axis is cooled by the interaction with an optical cavity. (b) The Cavendish-like torsional balance use for the direct measurement of the gravitational coupling between m_t and m_s. The Newtonian force is modulated by changing the separation between the two masses.

(a) The levitated oscillator of Ref. [h] is a silica nanosphere of radius $R_S = 70\,\mathrm{nm}$ and density $\rho = 1850\,\mathrm{Kg}$. Let's consider a one dimensional harmonic oscillator with a trap frequency of $\omega_m/2\pi = 160\,\mathrm{kHz}$. At the beginning of the experiment, in absence of cooling, the oscillator is in a classical thermal state.

1) From the equipartition theorem calculate the position variance σ_x^2 and the associated displacement rms.

2) How does it compare to zero point fluctuations $x_{xpf} = \sqrt{\frac{\hbar}{2m\omega_m}}$?

3) Calculate the mean thermal occupation n_{th} in the high temperature limit.

4) The main sources of decoherence are due to thermal noise and photons recoil. The former is due to collisions with the background gas which generates a damping:

$$\gamma_{gas} \simeq 0.8 \frac{Pressure\,[mbar]}{\rho R_S} \tag{22}$$

The fluctuation dissipation theorem connects this damping to a force noise. Expressed in terms of a heating rate this is given by $\Gamma_{th} = \gamma_{gas} n_{th}$.

Photon recoil is due to scattering of the trapping field which generates a force $F \simeq \hbar k n_{sc} = P_{sc}/c$. Here, n_{sc} is the number of scattered photons and P_{sc} is the scattered power. As for the thermal noise this force is stochastic and its heating rate can be calculated to be:

$$\Gamma_{rec} = \frac{2}{5} \frac{\omega_l}{\omega_m} \frac{P_{sc}}{m\,c^2} \tag{23}$$

where ω_l is the trapping field angular frequency.

Assuming a pressure of 10^{-7} mbar and a scattered power $P_{sc} = 10\,\mu\mathrm{W}$ ($\lambda = 1064\,\mathrm{nm}$) calculate Γ_{th} and Γ_{rec}.

5) In absence of cooling the evolution of the mean energy can be predicted by the following equation (Fokker-Planck):

$$\dot{n} = -\gamma_{gas} n + \sum_i \Gamma_i. \tag{24}$$

In the short time scale limit calculate the average number of oscillations in the time it take for n to increase by one assuming $n(0) = n_o \ll \sum_i \Gamma_i/\gamma_{gas}$.

(b) The torsion balance in Fig. 4 can be described as an harmonic oscillator with a resonance frequency $\omega_m/2\pi = 3.6\,\mathrm{mHz}$ and mechanical quality factor $Q = \omega_m/\gamma = 5$. The Newtonian force between the test mass $m_t \simeq 90\,\mathrm{mg}$ and the source mass $m_s \simeq 90\,\mathrm{mg}$ is modulated by changing the separation of the mass centers from 2.5 mm to 5.8 mm.

1) Calculate the expected force at the smallest separation;

2) To get an idea of the effect of the surrounding gravitational disturbances calculate how far has to be an experimenter (say 80 Kg) to generate a similar force. What about a nearby train (say a carriage weighting 15000 Kg)?

3) Assuming the oscillator is thermal the minimum detectable rms force is given by $F_{min} = \sqrt{4 k_B T m_{eff} \gamma BW}$, where T is the temperature of the environment, k_B is the Boltzman constant, $m_{eff} \simeq 2m_t$ is the effective mass of the torsion balance, γ the damping and BW is the measurement bandwith. Assuming a measurement bandwidth of 0.1 mHz (roughly 3 hours) what is the signal to noise ratio?

4) Among the disturbances that have to be considered the Casimir force is one of the most problematic for small separations. For a metallic sphere near a conductive plane the Casimir force is given by:

$$F_{cas} = \frac{R_S k_B T \zeta(3)}{8 d_s^2} \tag{25}$$

where $\zeta(3) \simeq 1.2$ is the Riemann Zeta function R_s is the sphere radius and $d_s = 250\,\mu\mathrm{m}$ is the sphere-Faraday shield separation. One can verify that for the current configuration F_{cas} is negligible when compared to the gravitational attraction.

However, it can become dominant for future experiments aiming to replicate the measurement with test masses approaching the Planck mass ($m_p \simeq 2.2 \times 10^{-8}\,\mathrm{Kg}$). Assuming a symmetric configuration, masses separation $r = 4R_S$ and $d_s = 2R_S$ calculate the radius at which the Casimir force equals the Newtonian force (gold density $\rho = 19000\,\mathrm{Kg/m^3}$). How does the resulting mass compare to the Planck mass?

Constants:
$k_B = 1.38 \cdot 10^{-23}\,\mathrm{J/K}$
$\hbar = 1.05 \cdot 10^{-34}\,\mathrm{J\,s}$
$G = 6.67 \cdot 10^{-11}\,\mathrm{N\,m^2/kg^2}$

References:
[h] U. Delic et al, Science 367, 892 (2020)
[i] T. Westphal et al, Nature 591, 225 (2021)