Quantum Field Theory – Lecture 3

Our aim now is to find real $(\phi^* = \phi)$ solutions to the Klein-Gordon equation,

$$(\partial^2 + m^2)\phi = 0 .$$

Fourier-transform method

We have the operator $\partial^2 + m^2$ that annihilates $\phi(x)$. A standard way to find the solutions in such situations is to use Fourier space. Define

real space field $\rightarrow$ $$\phi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \phi(t,\vec{k})\, e^{i\vec{k}\cdot\vec{x}}$$

momentum space field $\rightarrow$ $$\phi(t,\vec{k}) = \int d^3x\, \phi(t,\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}$$

Now go to the Klein-Gordon equation and compute:

$$\int \frac{d^3k}{(2\pi)^3} \left(\partial_t^2 - \vec{\nabla}^2 + m^2\right) \phi(t,\vec{k})\, e^{i\vec{k}\cdot\vec{x}} = 0 \Rightarrow$$

$$\Rightarrow \int \frac{d^3k}{(2\pi)^3} \left[\ddot{\phi}(t,\vec{k}) - (i\vec{k})\cdot(i\vec{k})\,\phi(t,\vec{k}) + m^2\phi(t,\vec{k})\right] e^{i\vec{k}\cdot\vec{x}} = 0$$

$$\Rightarrow \int \frac{d^3k}{(2\pi)^3} \left[\ddot{\phi}(t,\vec{k}) + \underbrace{(\vec{k}^2 + m^2)}_{\omega_{\vec{k}}^2}\, \phi(t,\vec{k})\right] e^{i\vec{k}\cdot\vec{x}} = 0$$

We can satisfy this equation if

$$\ddot{\phi}(t,\vec{k}) + \omega_{\vec{k}}^2\, \phi(t,\vec{k}) = 0$$

i.e. if $\phi(t,\vec{k})$ solves the differential equation for simple harmonic motion with frequency $\omega_{\vec{k}}$. Such solutions take the form

$$\phi(t,\vec{k}) = \phi(0,\vec{k})\, e^{\pm i\omega_{\vec{k}} t}$$

The naive solution is

$$\phi_{naive}(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \phi(0,\vec{k})\, e^{\pm i\omega_{\vec{k}} t}\, e^{i\vec{k}\cdot\vec{x}}$$

$k^0 = \omega_{\vec{k}} = +\sqrt{\vec{k}^2+m^2}$

$$\to \int \frac{d^3k}{(2\pi)^3}\, \phi(0,\vec{k})\, e^{-ik\cdot x}$$

where we chose the negative sign in $e^{\pm i\omega_{\vec{k}} t}$ and wrote $e^{-ik\cdot x} = e^{-i\omega_{\vec{k}} t + i\vec{k}\cdot\vec{x}}$. However now our naive solution is not real, but that is easy to fix since, for any complex number z, $z + z^*$ is real:

$$\phi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\left[\phi(0,\vec{k})\, e^{-ik\cdot x} + \phi^*(0,\vec{k})\, e^{ik\cdot x}\right]$$

In order to guarantee Lorentz invariance of the integration measure we normalise as follows:

transform like Lorentz scalars

$$\phi(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, \frac{1}{2\omega_{\vec{k}}}\left(a_{\vec{k}}\, e^{-ik\cdot x} + a^*_{\vec{k}}\, e^{ik\cdot x}\right).$$

This is our solution to the Klein-Gordon equation and now we want to quantise it.

Quantisation of single SHO

In classical mechanics

nothing to do with k above

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2, \qquad k = m\omega^2.$$

In the canonical approach to quantisation we proceed as follows:

(i) promote p, x, H to operators $\hat{p}, \hat{x}, \hat{H}$

(ii) impose commutation relations $(\hbar = 1)$

$$[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0, \quad [\hat{x}, \hat{p}] = i$$

(iii) in the Schrödinger picture the state $|\psi(t)\rangle$ evolves according to

$$\hat{H}\,|\psi(t)\rangle = i\frac{\partial}{\partial t}\,|\psi(t)\rangle.$$

There is a different apprach that is more useful as it generalises to QFT, namely the ladder operator approach. Here we write the Hamiltonian in the form

$$\hat{H} = \omega\,(\hat{a}^{+}\hat{a} + \tfrac{1}{2})$$

where

$$\hat{a} = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\,\hat{x} + \frac{i}{\sqrt{m\omega}}\,\hat{p}\right)$$

$$\hat{a}^{+} = \frac{1}{\sqrt{2}}\left(\sqrt{m\omega}\,\hat{x} - \frac{i}{\sqrt{m\omega}}\,\hat{p}\right).$$

Since $[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$ and $[\hat{x}, \hat{p}] = i$ we can show that

$$[\hat{a}, \hat{a}^{+}] = 1$$

$$[\hat{H}, \hat{a}] = -\omega\hat{a}$$

$$[\hat{H}, \hat{a}^{+}] = \omega\hat{a}^{+}$$

Now if we have an eigenstate $|n\rangle$ of the Hamiltonian, then

$$\begin{aligned}\hat{H}\,(\hat{a}^{+}|n\rangle) &= [\hat{H}, \hat{a}^{+}]\,|n\rangle + \hat{a}^{+}\hat{H}\,|n\rangle \\ &= \omega\,\hat{a}^{+}|n\rangle + E_n\,\hat{a}^{+}|n\rangle \\ &= (\omega + E_n)\,\hat{a}^{+}|n\rangle\end{aligned}$$

and thus we may denote

$$\hat{a}^\dagger |n\rangle \propto |n+1\rangle ,$$

where $|n+1\rangle$ is an eigenstate of $\hat{H}$ with energy $E_{n+1} = E_n + \omega$. Similarly,

$$\hat{H}\,(\hat{a}\,|n\rangle) = (E_n - \omega)\,\hat{a}\,|n\rangle$$

and

$$\hat{a}\,|n\rangle \propto |n-1\rangle, \quad E_{n-1} = E_n - \omega .$$

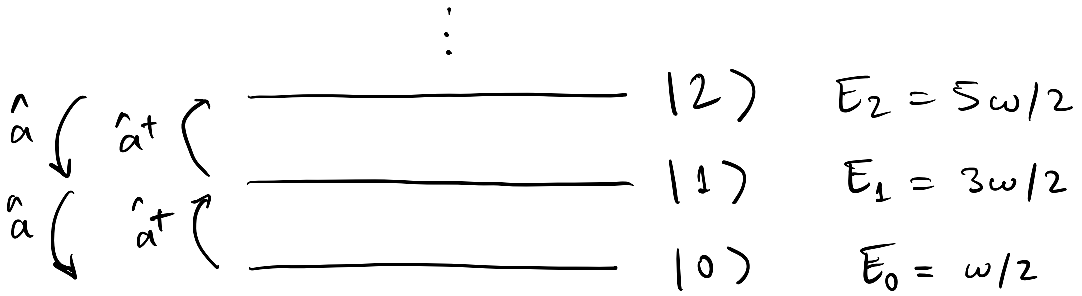
For any energy eigenstate $|n\rangle$, $\hat{a}^\dagger$ raises its energy by one unit of ω and $\hat{a}$ lowers it by one unit of ω. The ground state is the state $|0\rangle$ such that

$$\hat{a}\,|0\rangle = 0$$

we do not want negative energies

and then

$$\hat{H}\,|0\rangle = \omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right)|0\rangle = \tfrac{1}{2}\omega\,|0\rangle .$$



It turns out that all states can be built from $|0\rangle$:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle , \qquad \langle n|m\rangle = \delta_{nm} \text{ for any } m,n \geq 0.$$

Generalisation to multiple decoupled SHOs

Suppose we have N SHOs:

$$H = \sum_{i=1}^{N} \hat{H}_i , \qquad \hat{H}_i = \frac{\hat{p}_i^2}{2m} + \frac{1}{2} m\omega \hat{x}_i^2 .$$

same m, ω for all i

Now we write $\hat{H}_i = \omega\left(\hat{a}_i^+ \hat{a}_i + \frac{1}{2}\right)$ and these a's satisfy

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^+, \hat{a}_j^+] = 0, \qquad [\hat{a}_i, \hat{a}_j^+] = \delta_{ij}.$$

We denote states by

$$|n_1, n_2, \ldots, n_N\rangle = |n_1\rangle \otimes |n_2\rangle \otimes \cdots \otimes |n_N\rangle$$

where each of the N SHOs is in a state $|n_i\rangle$ independently of the others (product state). The operator $\hat{a}_i^+$ acts as follows:

$$\hat{a}_i^+ |n_1, n_2, \ldots, n_i, \ldots, n_N\rangle = |n_1, n_2, \ldots, n_i + 1, \ldots, n_N\rangle.$$

The vacuum state is that of all SHOs in their vacuum state:

$$a_i |0, 0, \ldots, 0\rangle = 0 \quad \text{for all } i = 1, \ldots, N.$$

Excited states are given by

$$|n_1, \ldots, n_N\rangle = \frac{(\hat{a}_1^+)^{n_1} \cdots (\hat{a}_N^+)^{n_N}}{\sqrt{n_1!} \cdots \sqrt{n_N!}} |0, \ldots, 0\rangle.$$

This is called the occupation number representation.

Summary

- Solutions to Klein-Gordon equation are linear superpositions of an infinite number of SHOs for each mode $\vec{k}$.
- We solved the SHO with raising/lowering (or creation/annihilation) operators.

Recap of Day 1

- Talked about what QFT is and why we use it.
- Fundamental degrees of freedom: fields
- We can write Lagrangians and Hamiltonians (densities) with fields
- Free massive scalar field: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

  $\mathcal{H} = \frac{1}{2}(\pi^2 + \vec{\nabla}\phi\cdot\vec{\nabla}\phi + m^2\phi^2)$

  $\pi = \dot{\phi}$
- EOM: $(\partial^2 + m^2)\phi = 0$

  Klein – Gordon equation
- Solutions in Fourier space: each Fourier mode ($\vec{k}$ component) satisfies the simple harmonic motion equation with $\omega_{\vec{k}}^2 = \vec{k}^2 + m^2$.
- Quantised SHOs using raising / lowering operators.

* Invariance of integration measure

Using k^μ we may write the manifestly Lorentz invariant

$$M = \frac{d^4k}{(2\pi)^4}\, 2\pi\, \underbrace{\delta^{(4)}(k^2 - m^2)}_{\text{to make sure that K-G eq. is satisfied}}\, \theta(k^0)$$

enforces $k^0 > 0$

Then,

$$M = \frac{d^3k\,dk^0}{(2\pi)^3}\,\delta^{(4)}\left((k^0-\omega_{\vec{k}})(k^0+\omega_{\vec{k}})\right)\theta(k^0)$$

$$= \frac{d^3k\,dk^0}{(2\pi)^3}\,\frac{1}{2k^0}\left(\delta^{(4)}(k^0-\omega_{\vec{k}})\,\theta(k^0) + \delta^{(4)}(k^0+\omega_{\vec{k}})\,\theta(k^0)\right)$$

$$\delta(f(x)) = \sum_{x\,|\,f(x)=0} \frac{1}{|f'(x)|}\,\delta(x)$$

$$= \frac{d^3k}{(2\pi)^3}\,\frac{1}{2\omega_{\vec{k}}},$$

which is the normalisation we used above.