Finite-size scaling of Lee-Yang zeros and its application to 3-state Potts model and Heavy-quark QCD

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Lattice 2024, Liverpool Univ., Aug. 2, 2024

Approaches to Nonzero μ_B

- Monte Carlo
  - Taylor expansion
  - Imaginary chemical potential
- Complex Langevin
- Lefschetz thimble

Lee-Yang edge singularity

D.A.Clarke, et al.
arXiv:2405.10196[hep-lat]

- 3d-Z(2) Lee-Yang edge singularity

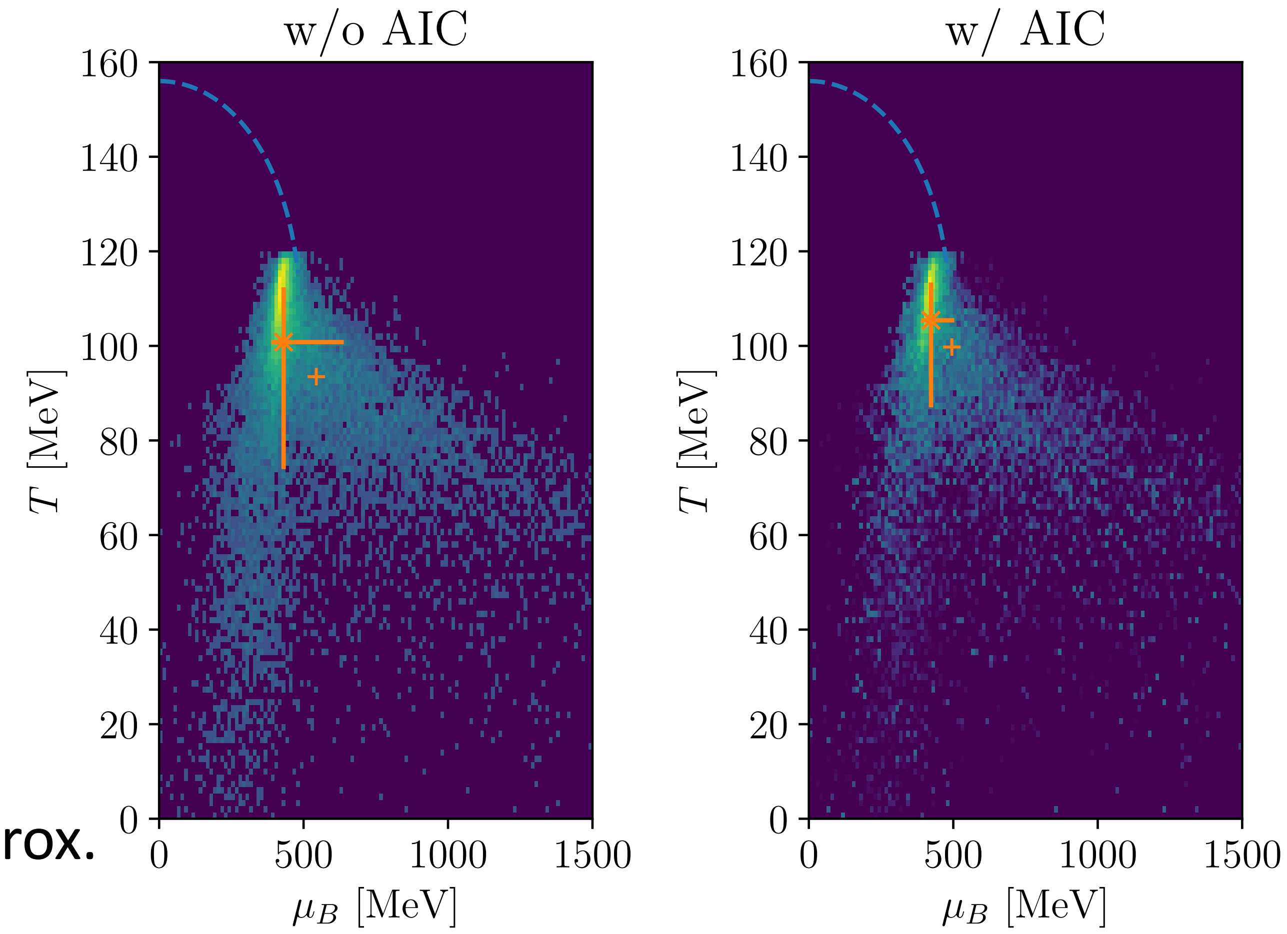
Karsch *et al., PRD 109,014508* (2023)

- 3d-O(N) Lee-Yang edge singularity

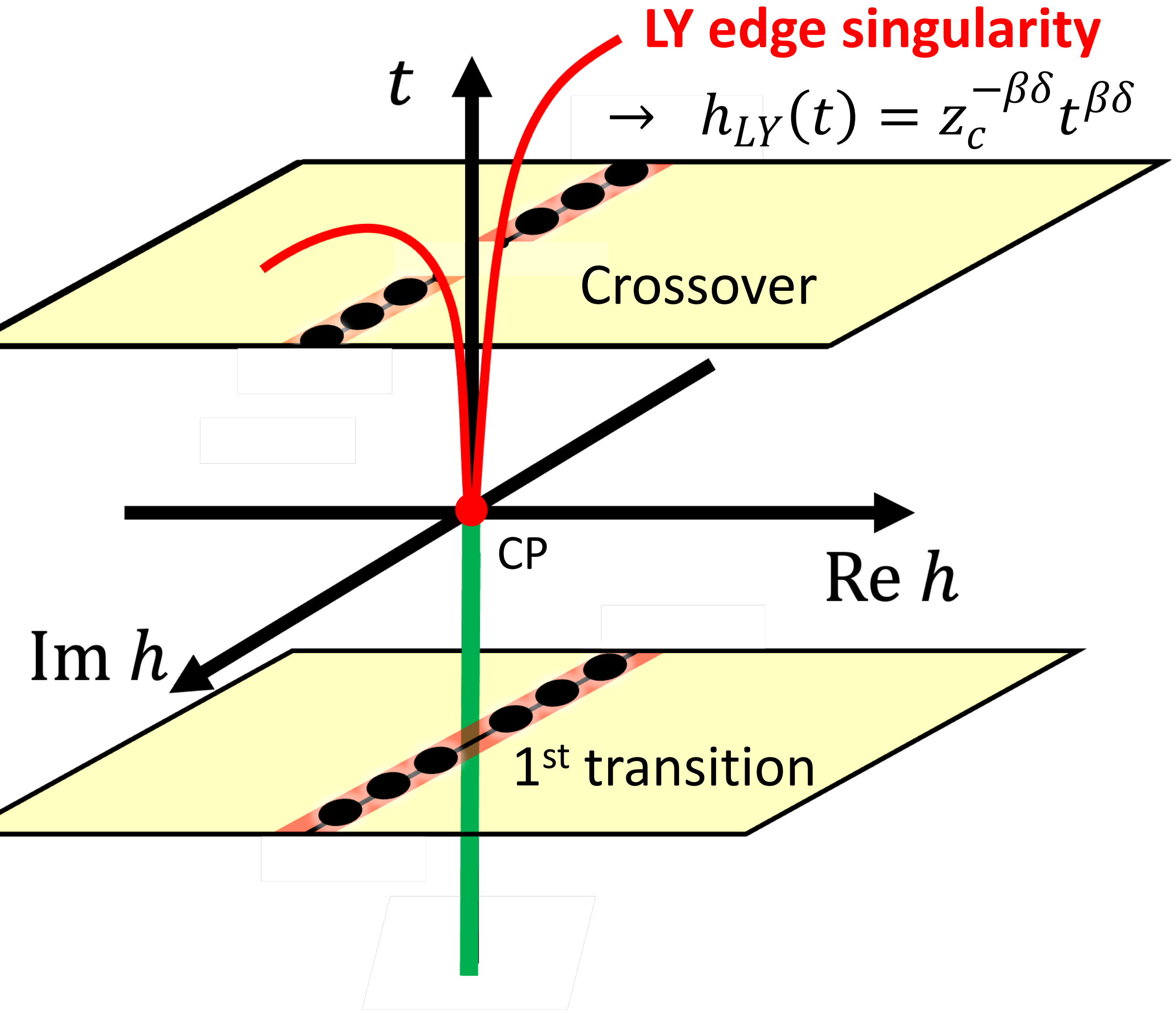
Skokov et al., PRD 107,116013 (2023)

- QCD Roberge-Weiss, Chiral with Pade approx.

Schmidt et al., PRD 105,034513 (2022)



Lee-Yang Zeros (LYZ) of Ising Model



Lee-Yang Zeros
= Zeros of Z in complex h

Lee-Yang circle theorem

LYZ are located only on the Im-axis.

$V \rightarrow \infty$

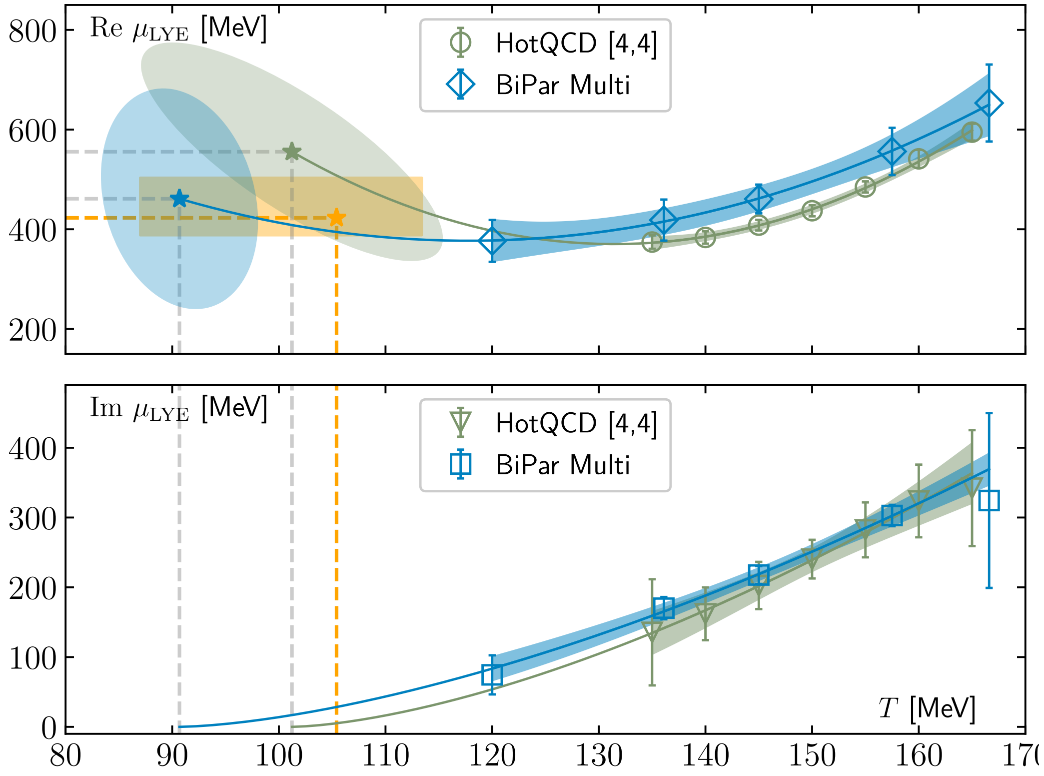
LYZ becomes denser

- $t \leq 0$, LYZ intersect with Re-axis
- $t > 0$, LYZ are away from Re-axis

→Edge of LYZ = LY edge singularity

QCD-CP

D.A.Clarke, et al. arXiv:2405.10196[hep-lat]



Assumption:

1st LYZ = LY edge singularity

$$\begin{cases} \mathrm{Re}\ \mu_{\mathrm{LYE}} = \mu_B^{\mathrm{CEP}} + c_1 \Delta T + c_2 \Delta T^2 \\ \mathrm{Im}\ \mu_{\mathrm{LYE}} = c_3 \Delta T^{\beta\delta} \end{cases}$$

Fit Lattice data

$$\begin{cases} \mu^{\mathrm{CEP}} = 422^{+80}_{-35}\mathrm{MeV} \\ T^{\mathrm{CEP}} = 105^{+8}_{-18}\mathrm{MeV} \end{cases}$$

Purpose

- ◆ **Explore the properties of LYZ in finite V via FSS**
- ◆ **Propose a new method for locating CP from results on finite V**

Finite-Size Scaling

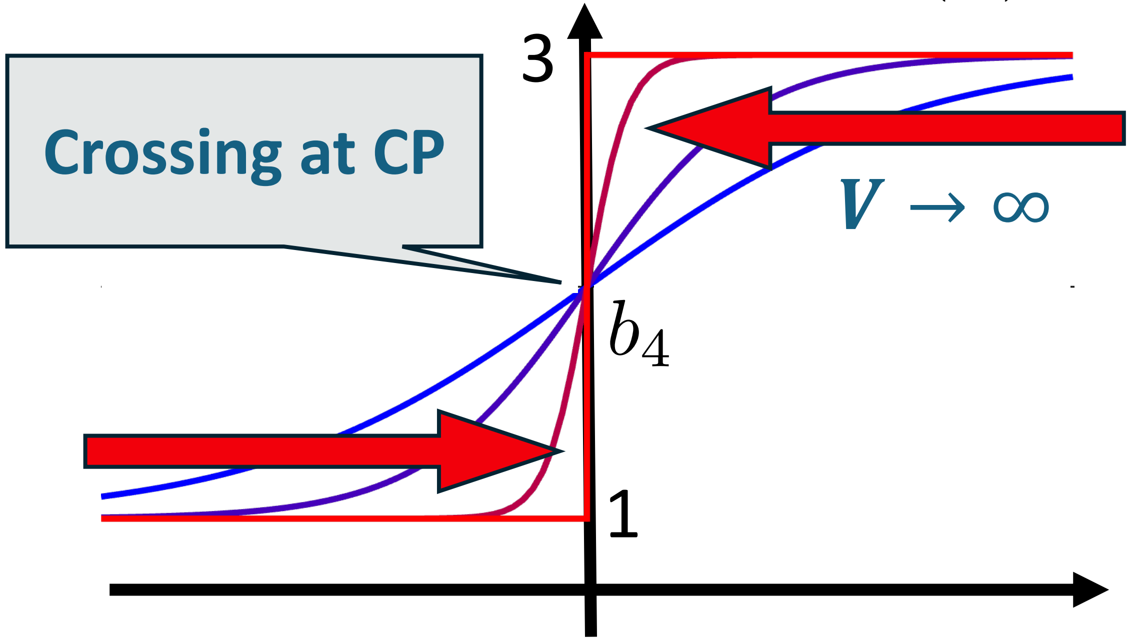
FSS of free energy in finite *V*

$$F(t,h,L^{-1}) = \tilde{F}(L^{y_t}t, L^{y_h}h)$$

Binder Cumulant

$$B_4(t,h,L^{-1}) = \frac{\partial_h^4 \tilde{F}(L^{y_t}t,(L^{y_h}h)}{(\partial_h^2 \tilde{F}(L^{y_t}t,(L^{y_h}h))^2} + 3$$

$$B_4(t,0,L^{-1}) = b_4 + ctL^{y_t} + \mathcal{O}\left(t^2\right)$$



Partition function

$$Z(t,h,L^{-1}) = \prod_i \left(h - h_{\rm LY}^{(i)}(t,L^{-1})\right)$$

$$= \prod_i \left(L^{y_h}h - \tilde{h}_{\rm LY}^{(i)}(L^{y_h}t)\right)$$

FSS of LYZ

$$L^{y_h}h = \tilde{h}_{\rm LY}^{(i)}(L^{y_t}t)$$

Develop a useful relation
to locate CP like Binder cumulant.

LYZ in 3d-Ising Model

$$L^{y_h} h = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)$$

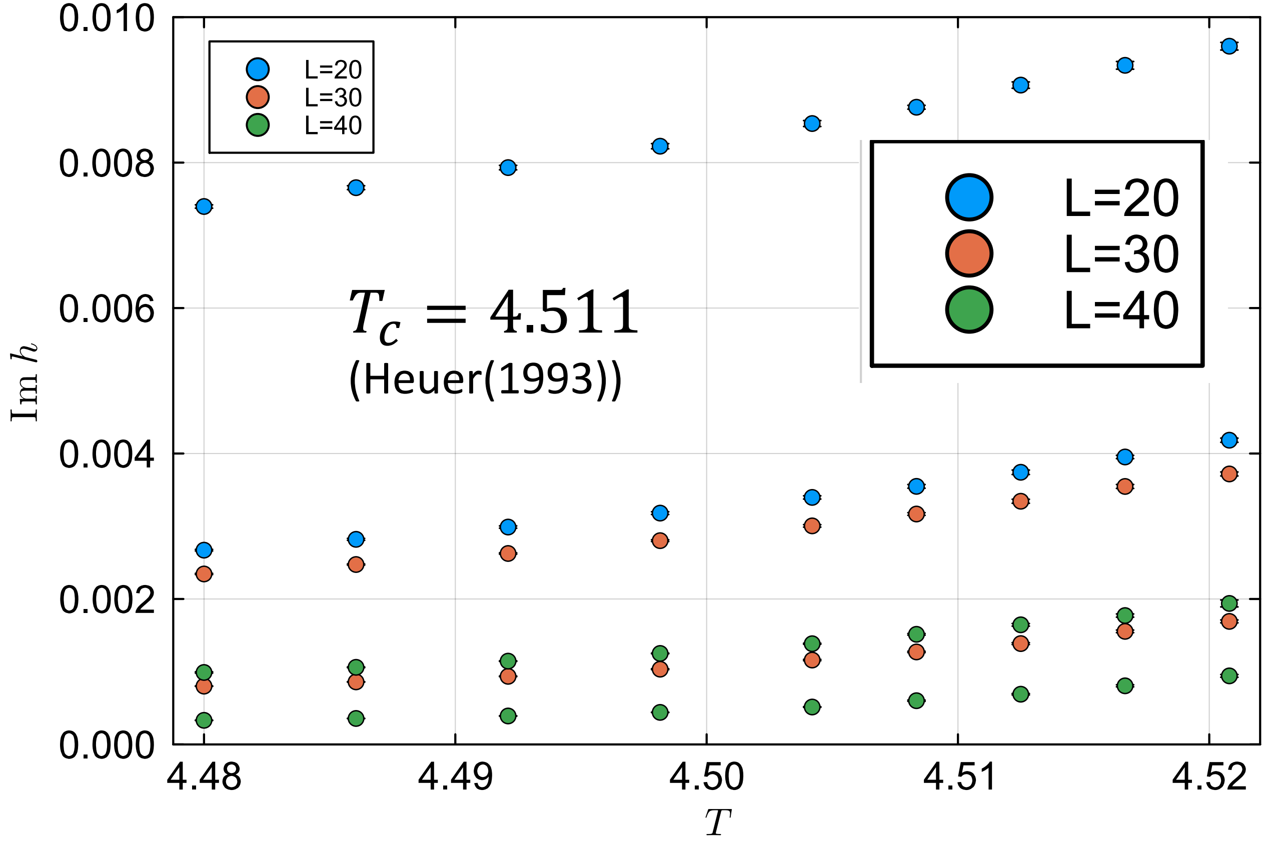
Numerical Analysis of 3d-Ising Model

$$H = -\sum_{i,j} s_i s_j - h \sum_i s_i$$

Setup:

- Monte Carlo Simulation
- Reweighting for imaginary h
- #measurement = 10^5
- Volume V= $\mathrm{L}^3 = 20^3, 30^3, 40^3$

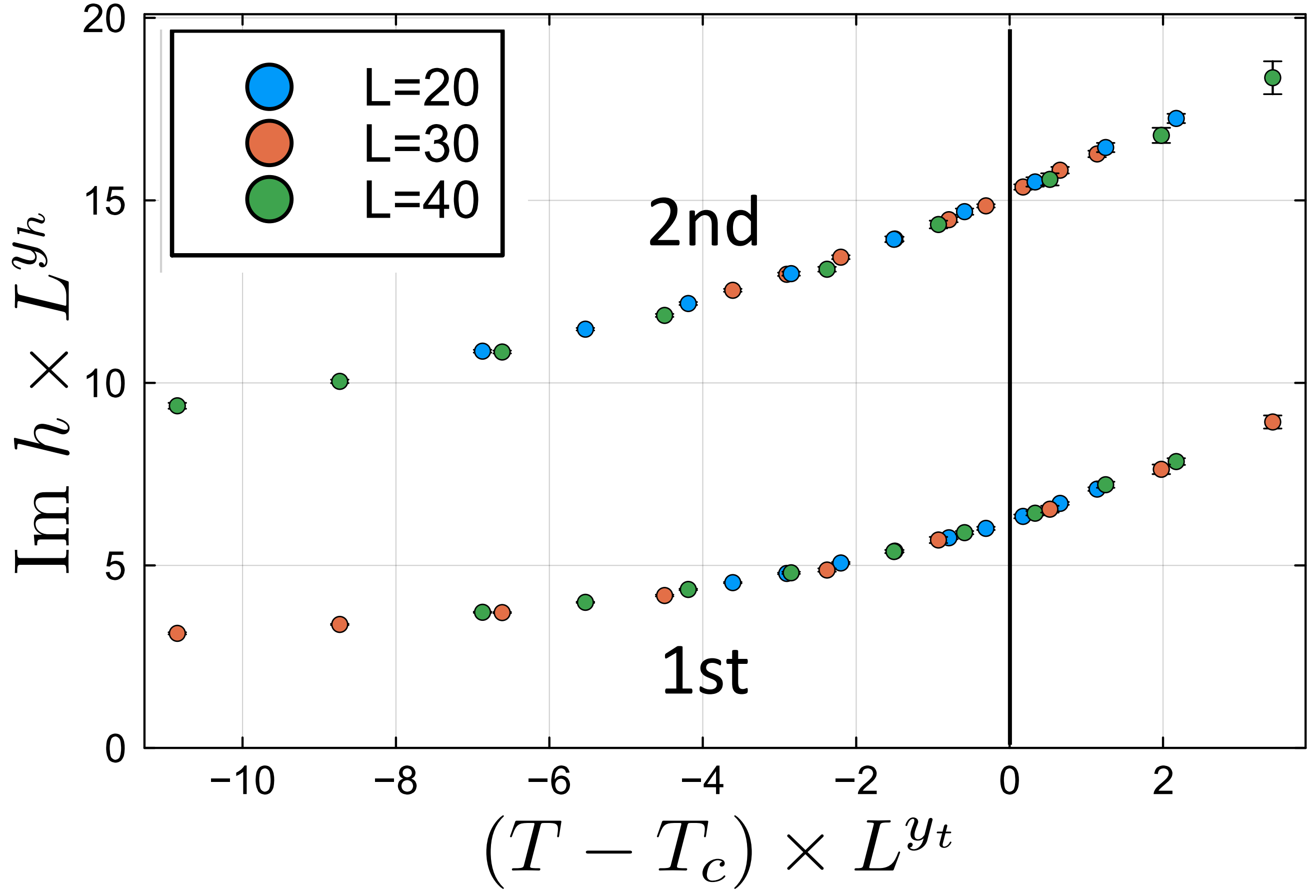
1st and 2nd LYZ



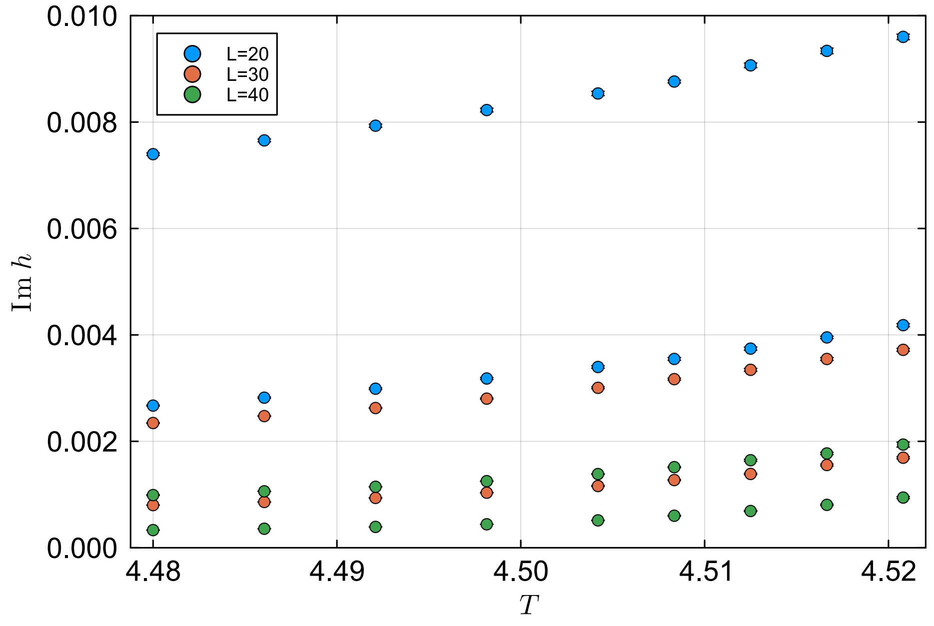
Rescaling of LYZ

$$L^{y_h} h = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)$$

Rescaled result



1st and 2nd LYZ

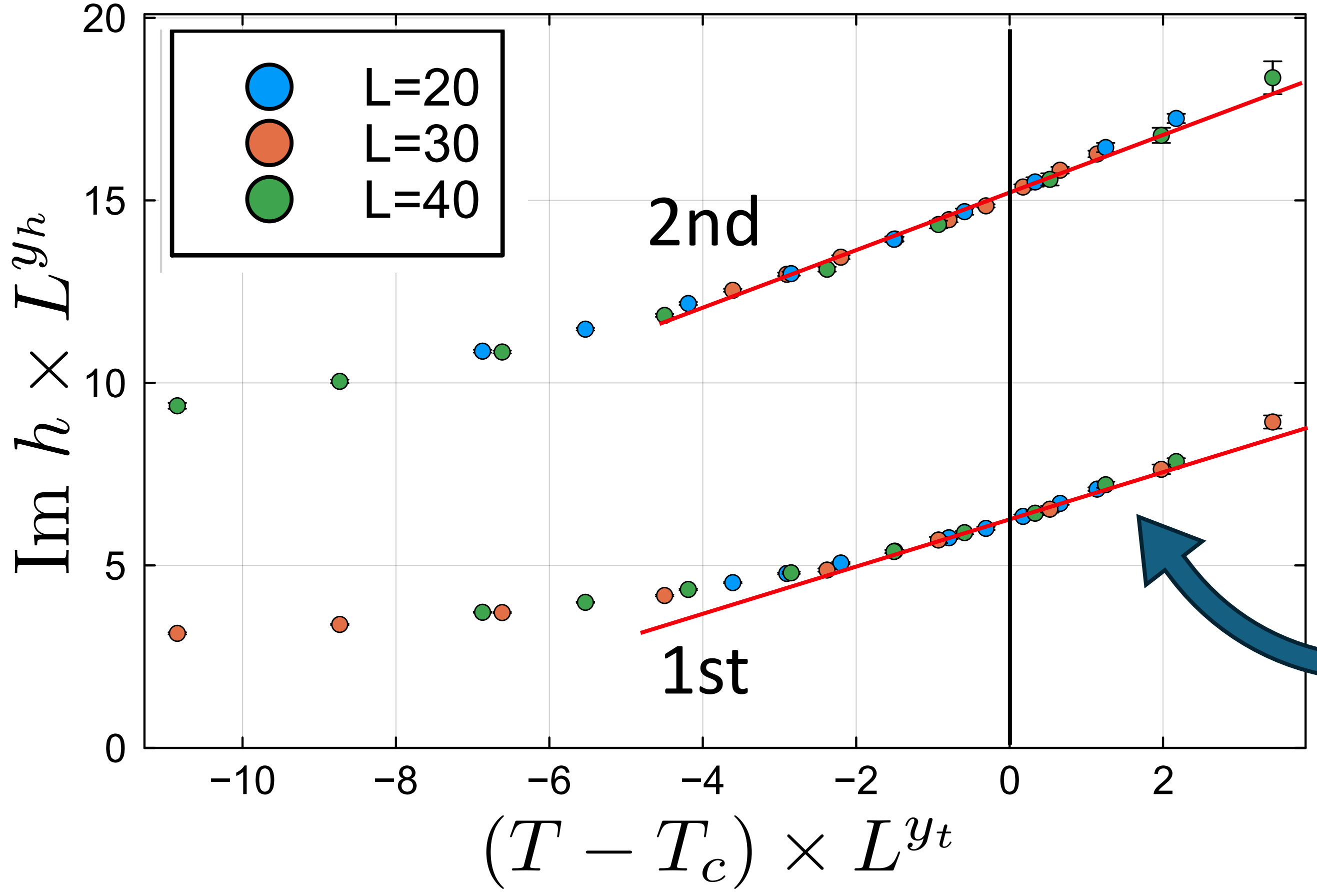


FSS of LYZ is numerically confirmed!

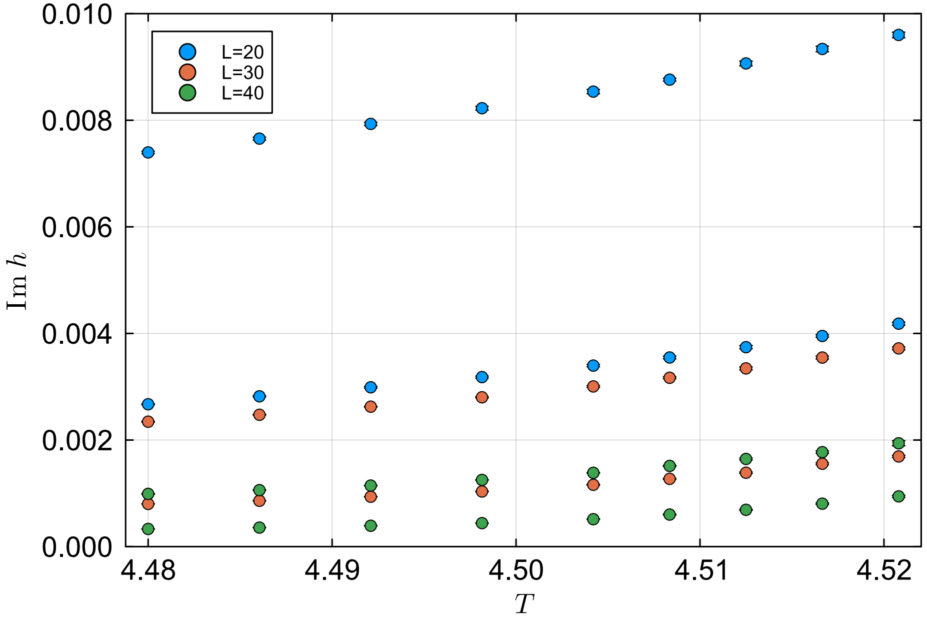
Rescaling of LYZ

$$L^{y_h} h = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)$$

Rescaled result



1st and 2nd LYZ



Linear approximation around CP

$$h^{(n)}_{\mathrm{LY}}(tL^{y_t}) \simeq i(X_n + Y_n t L^{y_t})$$

FSS of LYZ is numerically confirmed!

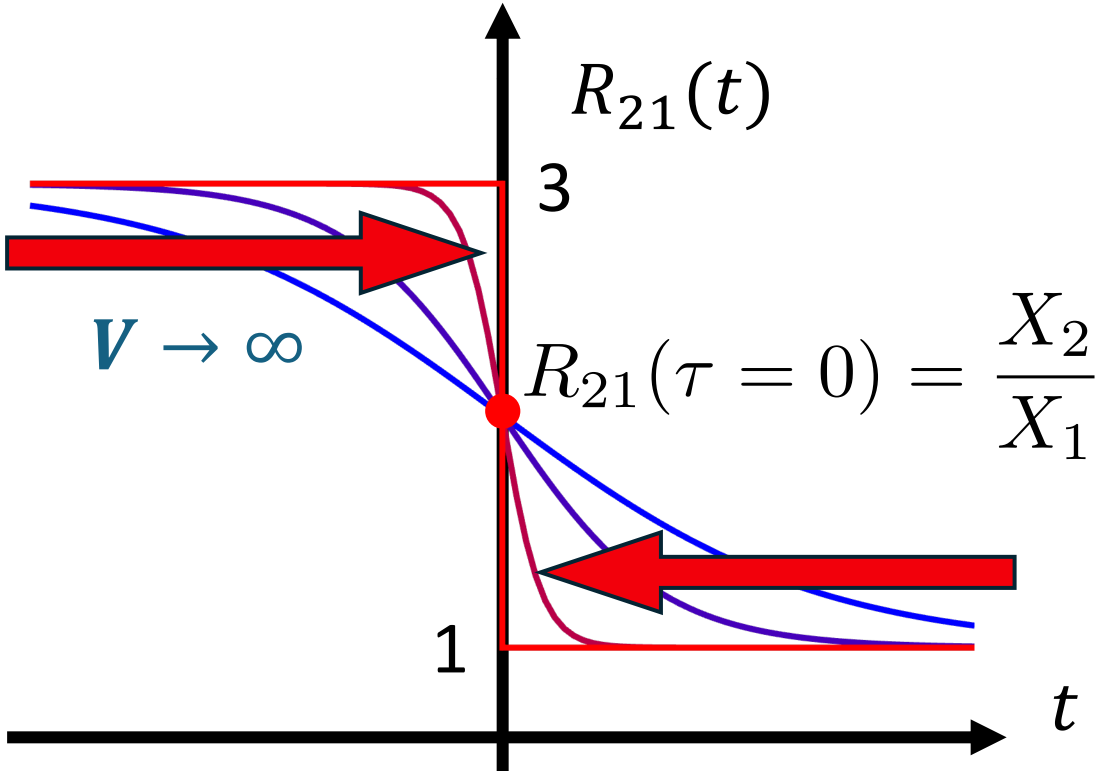
LYZ Ratio

FSS & Linear approximation

$$h^{(n)}L^{y_h} = \tilde{h}^{(n)}_{\mathrm{LY}}(tL^{y_t}) \simeq i(X_n + Y_n t L^{y_t})$$

$$R_{nm}(t) \equiv \frac{h^{(n)}(t)}{h^{(m)}(t)} = \frac{X_n}{X_m}\left(1 + \left(\frac{Y_n}{X_n} - \frac{Y_m}{X_m}\right)tL^{y_t} + \mathcal{O}(t^2)\right)$$

$$R_{21}(t) = \begin{cases} 3 & t \to -\infty \\ X_2/X_1 & t = 0 \\ 1 & t \to \infty \end{cases}$$



$\boldsymbol{R_{21}(0)}$ is V independent

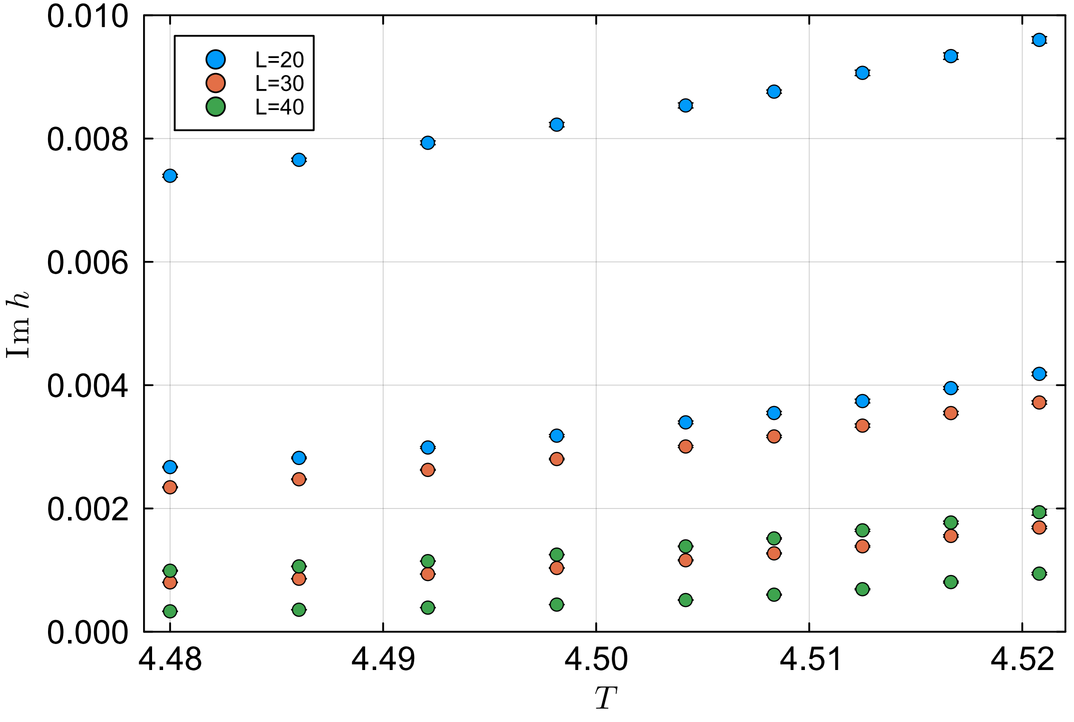
→ Useful for the CP search!

Cf. Binder cumulant $B_4(t = 0) = b_4$

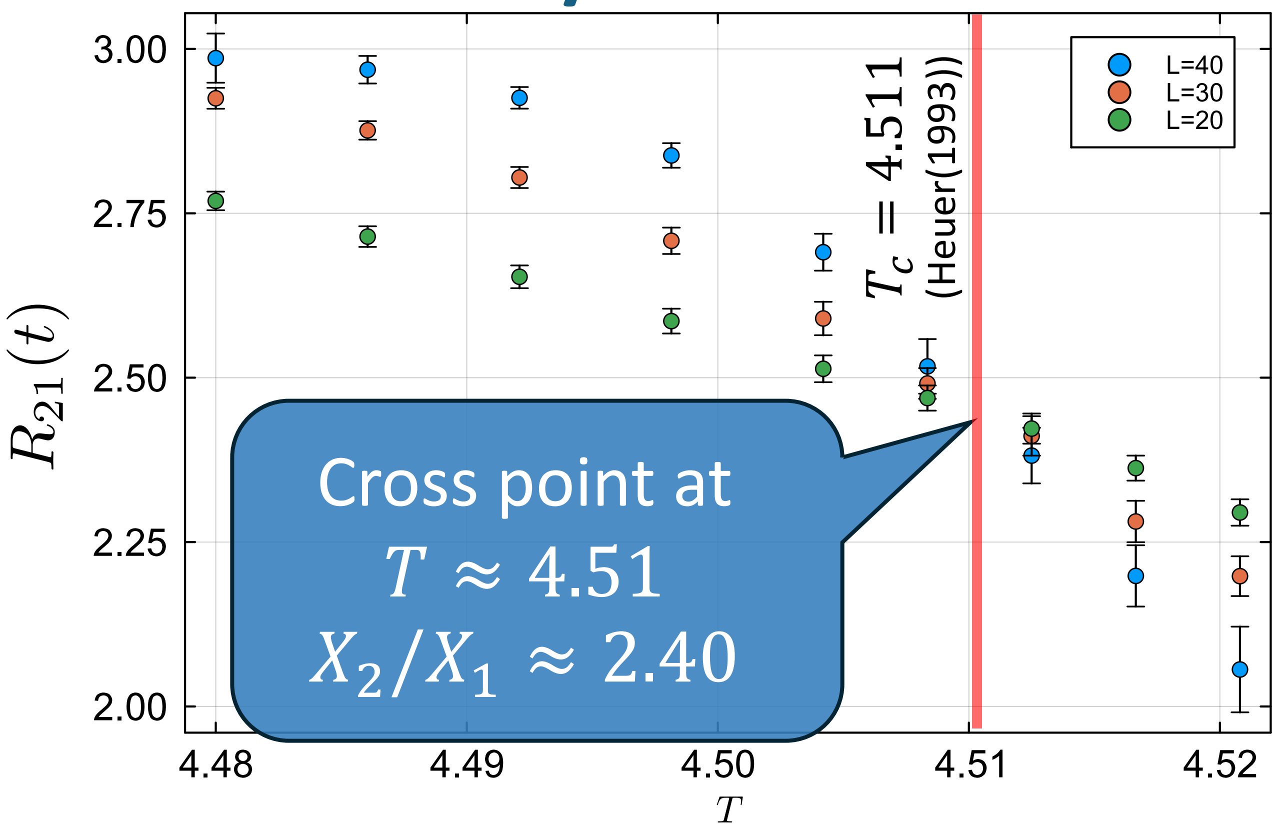
Numerical Test of LYZ Ratio

$$R_{21}(t) \equiv \frac{h^{(2)}(t)}{h^{(1)}(t)} = \frac{X_2}{X_1}\left(1 + \left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right) t L^{y_t} + \mathcal{O}(t^2)\right)$$

1st and 2nd LYZ

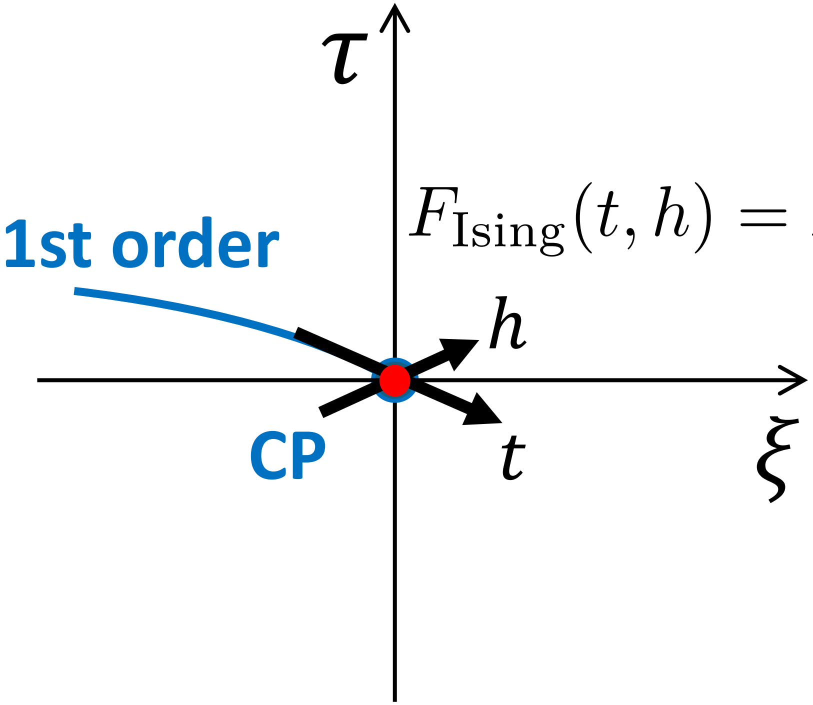


2^{nd} / 1^{st} LYZ Ratio



CP in General Systems

Search for LYZ in $\xi \in \mathbb{C}$ for $\tau \in \mathbb{R}$



$$F_{\text{Ising}}(t,h) = F(\tau,\xi)$$

Linear Approximations

$$\begin{pmatrix} t \\ h \end{pmatrix} = A \begin{pmatrix} \tau \\ \xi \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau \\ \xi \end{pmatrix}$$

$$h^{(n)} L^{y_h} = \tilde{h}^{(n)}_{\text{LY}}(tL^{y_t}) \simeq i(X_n + Y_n t L^{y_t})$$

$$\begin{cases} \xi_R^{(n)} L^{y_t} = & -\dfrac{a_{21}}{a_{22}} \tau L^{y_t} + \mathcal{O}\left(L^{2(y_t - y_h)}\right) \\ \xi_I^{(n)} L^{y_h} = & \dfrac{X_n}{a_{22}} + \dfrac{\det A \ \ Y_n}{a_{22}^2} \tau L^{y_t} + \mathcal{O}\left(L^{2(y_t - y_h)}\right) \end{cases}$$

Infinite Volume Limit

$$\begin{cases} \xi_R^{(n)} = & -\dfrac{a_{21}}{a_{22}} \tau \\ \xi_I^{(n)} = & \text{const.} \times \tau^{\beta\delta} \end{cases}$$

$$\begin{cases} \text{Re}\, \mu_{\text{LYE}} = \mu_B^{\text{CEP}} + c_1 \Delta T \\ \text{Im}\, \mu_{\text{LYE}} = c_3 \Delta T^{\beta\delta} \end{cases}$$

Consistent with Stephanov, PR**D73** (2006)

LYZ Ratio for General Systems

$$\begin{cases} \xi_R^{(n)} L^{y_t} = & -\frac{a_{21}}{a_{22}} \tau L^{y_t} + \mathcal{O}\left(L^{2(y_t - y_h)}\right) \\ \xi_I^{(n)} L^{y_h} = & \frac{X_n}{a_{22}} + \frac{\det A \; Y_n}{a_{22}^2} \tau L^{y_t} + \mathcal{O}\left(L^{2(y_t - y_h)}\right) \end{cases}$$

$$y_t - y_h = -0.894 \quad \text{for 3d-Z(2)}$$

$$C = \frac{\det A}{a_{22}} \left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1} \right), \quad D = \frac{a_{12}^2}{a_{22}^2} (Y_1^2 - Y_2^2)$$

$$R_{21}(\tau) = \frac{\xi_I^{(2)}(\tau)}{\xi_I^{(1)}(\tau)} = \frac{X_2}{X_1} \left(1 + C(\tau L^{y_t}) + \mathcal{O}(\tau^2)\right) \left(1 + D L^{2(y_t - y_h)} + \mathcal{O}\left(L^{4(y_t - y_h)}\right)\right)$$

Mixing from energy-like

General CP ($L \to \infty, \tau = 0$)

$$R_{21}(\tau = 0) = \frac{X_2}{X_1}$$

Equivalent

Ising model ($t = 0$)

$$R_{21}(t = 0) = \frac{h_{\mathrm{LY}}^{(2)}(0)}{h_{\mathrm{LY}}^{(1)}(0)} = \frac{X_2}{X_1}$$

- $R_{21}(0) = X_2/X_1$ does not depend on the mixing matrix A.
- X_2/X_1 is a specific to the universality class.

Comparison with Binder Cumulant Method

Binder Cumulant

$y_t - y_h = -0.894 \quad \text{for 3d-Z(2)}$

$$B_4(t,h,L^{-1}) = b_4(1+\tilde{c}tL^{y_t}+\mathcal{O}(t^2)) \times (1+dL^{y_t-y_h}+\mathcal{O}\left(L^{2(y_t-y_h)}\right))$$

Jin, *et al*. PR**D96** (2017)

LYZ Ratio

$$R_{21}(\tau) = \frac{X_2}{X_1}(1+C(\tau L^{y_t})+\mathcal{O}(\tau^2)) \times \left(1+DL^{2(y_t-y_h)}+\mathcal{O}\left(L^{4(y_t-y_h)}\right)\right)$$

Purely magnetic

Mixing from energy-like

- ◆ **Crossing analysis is applicable in both methods.**
- ◆ **Stronger suppression of mixing effect in the LYZ ratio.**

Numerical Confirmation

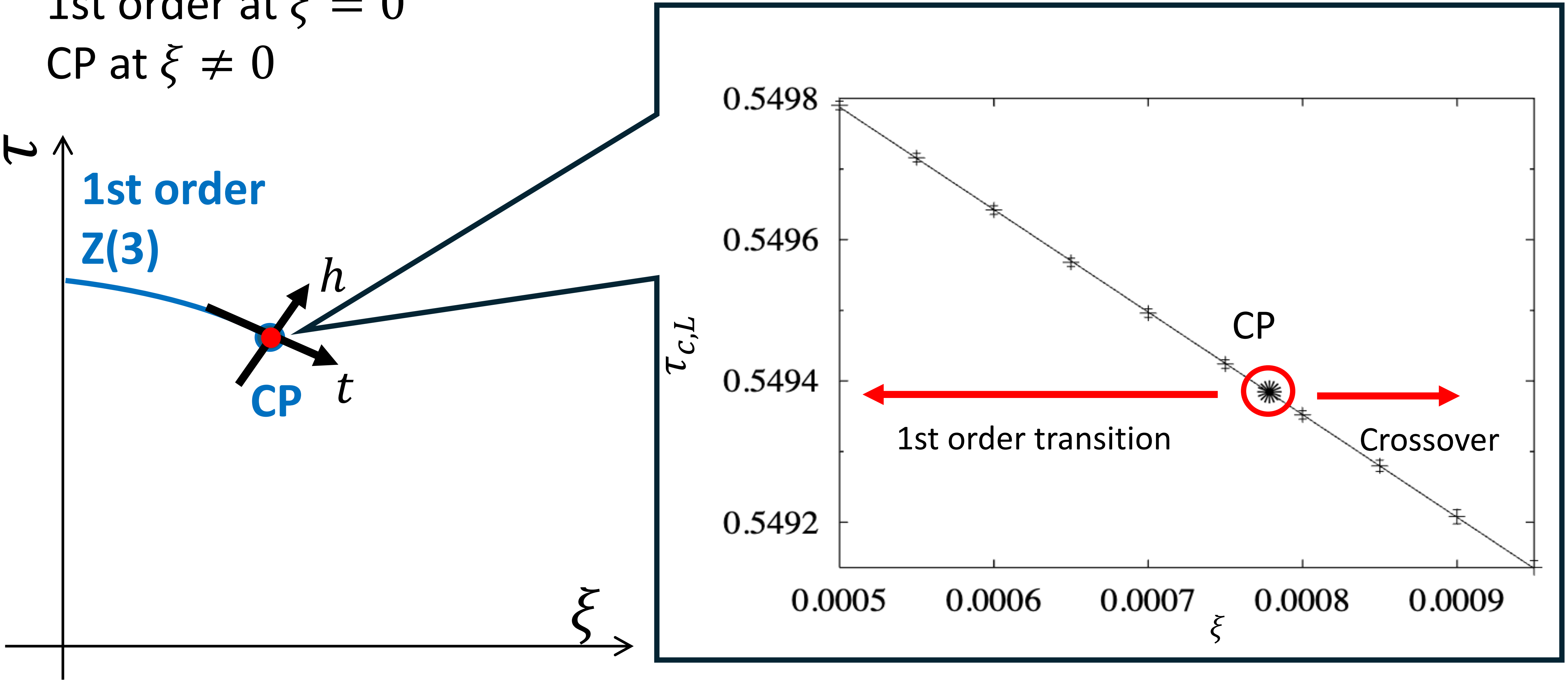
- 3d 3-State Potts model
- Heavy-Quark QCD

3d 3-state Potts model

Karsch, Stickan (2000) PLB, 488, 3-4

$$H = -\tau \sum_{i,j} \delta_{\sigma_i,\sigma_j} - \xi \sum_i \delta_{\sigma_i,1} \qquad (\sigma_i = 1,2,3)$$

- 1st order at $\xi = 0$
- CP at $\xi \neq 0$

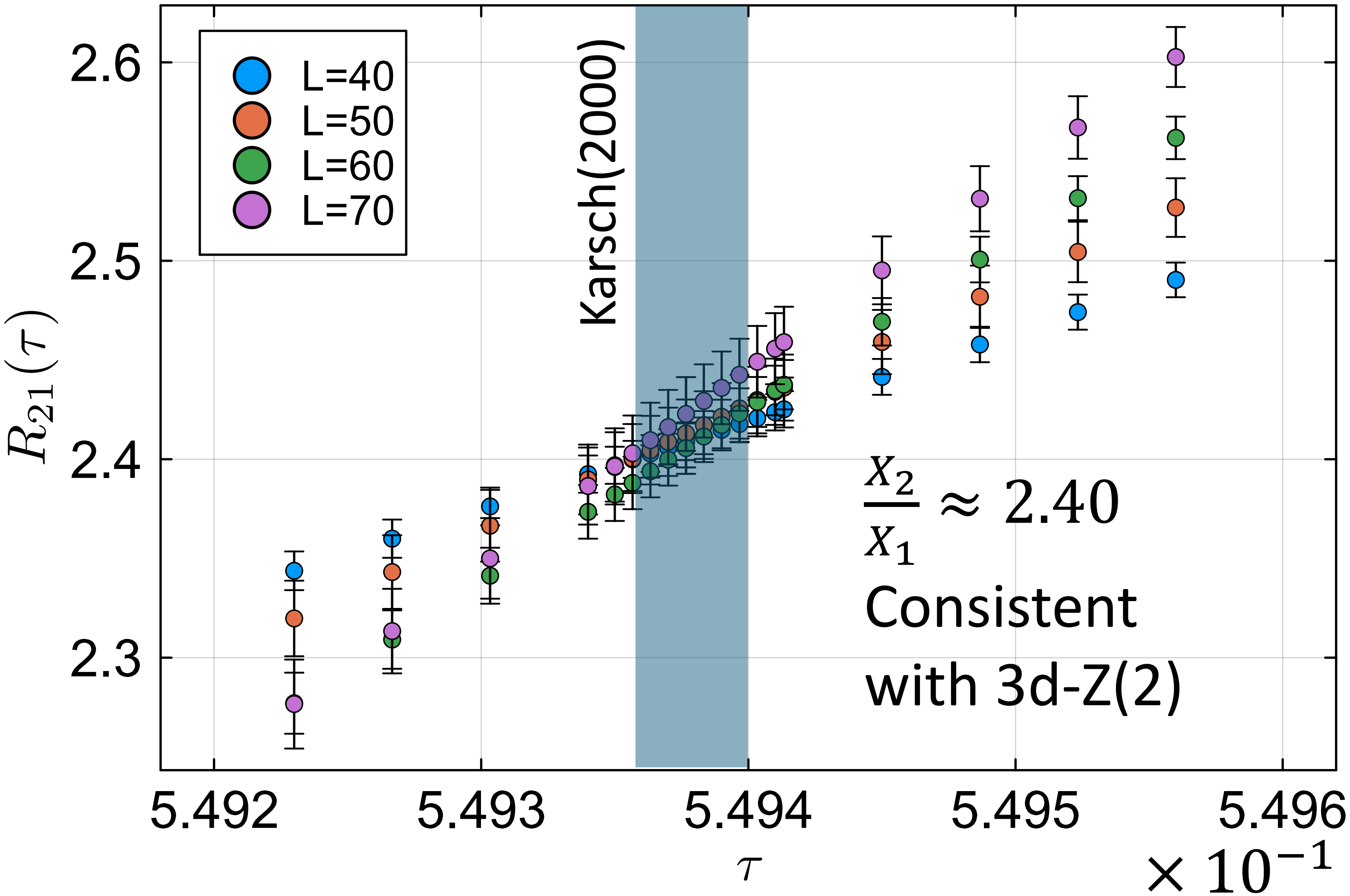


LYZ Ratio in 3d 3-state Potts Model

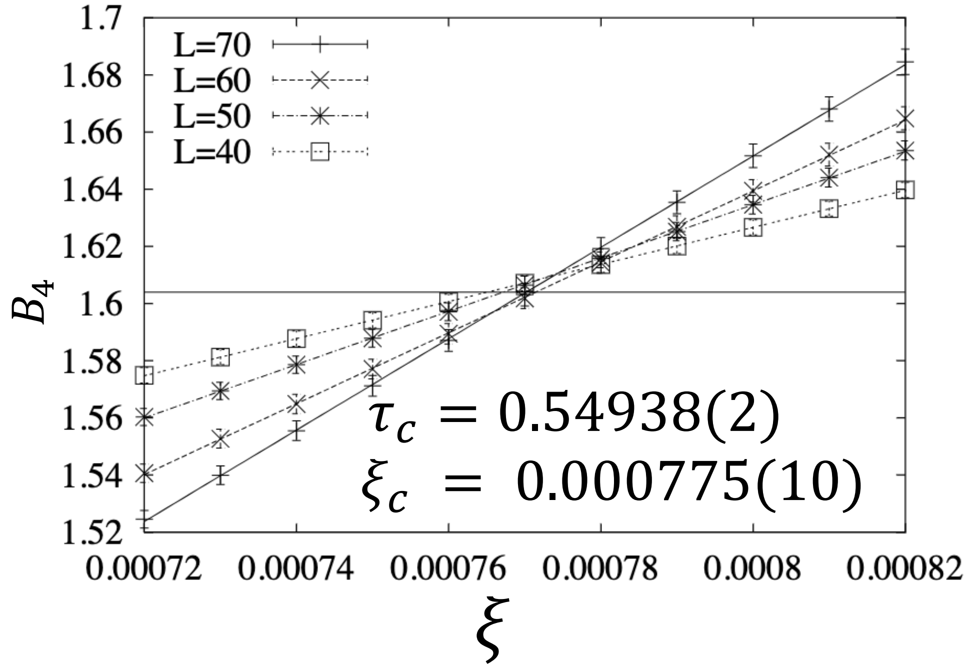
$V = 40^3, 50^3, 60^3, 70^3$

#measurement=4~7×10^5

2nd /1st LYZ Ratio



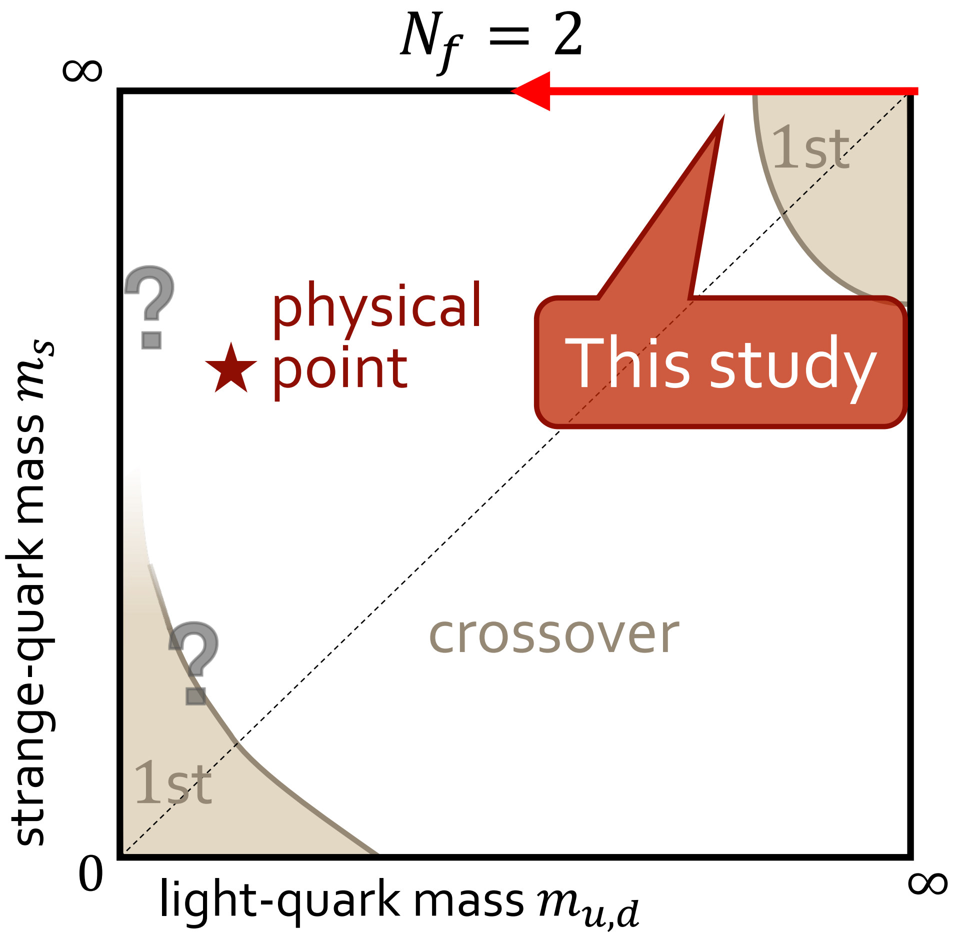
Binder cumulant (Karsch(2000))



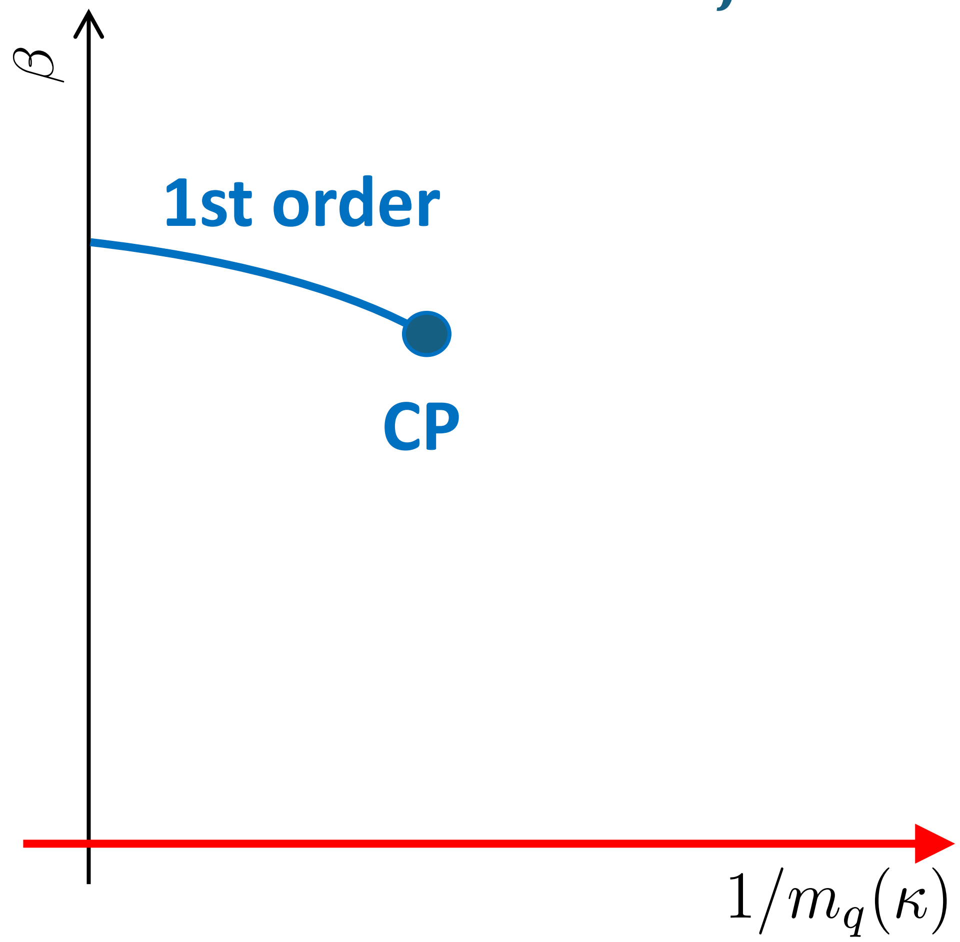
Order parameter $M = \sum_i \delta_{\sigma_i,1}$

Heavy-Quark QCD

Columbia Plot



Phase diagram for $N_f = 2$



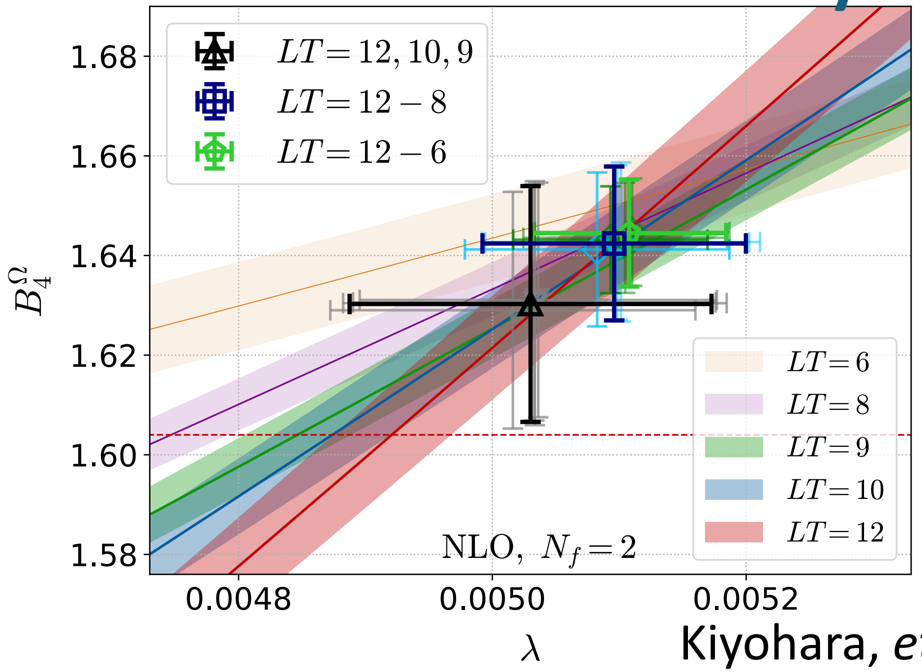
Heavy-Quark QCD setting

Kiyohara, *et al.* *PR***D104***(2021)*

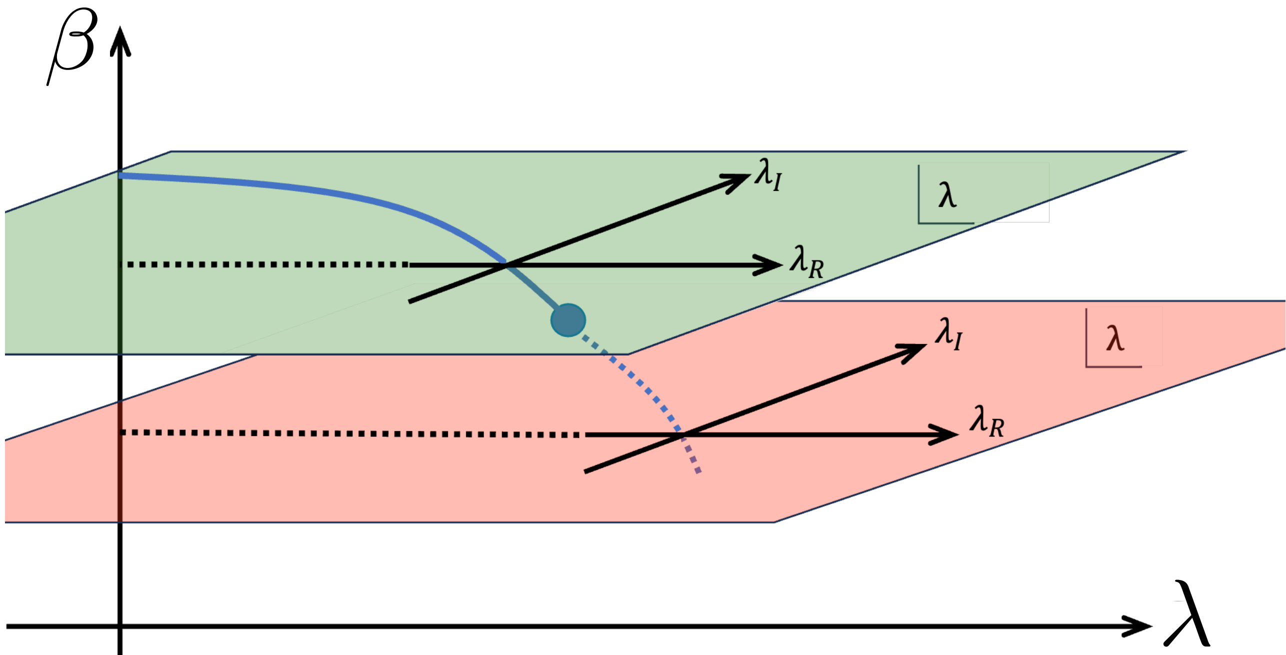
- With Hopping Paramar Expansion
- Lattice size : $N_s^3 \times (N_t = 4)$, aspect ratio $LT = N_s/N_t = 8,9,10,12$
- Each 600,000 measurements (same data set as Kiyohara, et al.)
- $\lambda = 64 N_c N_f \kappa^4$
- Fix $\beta \in \mathbb{R}$, searching for complex λ-plane

$$\kappa \sim \frac{1}{2am} \ \text{Hopping parameter}$$

Binder Cumulant analysis

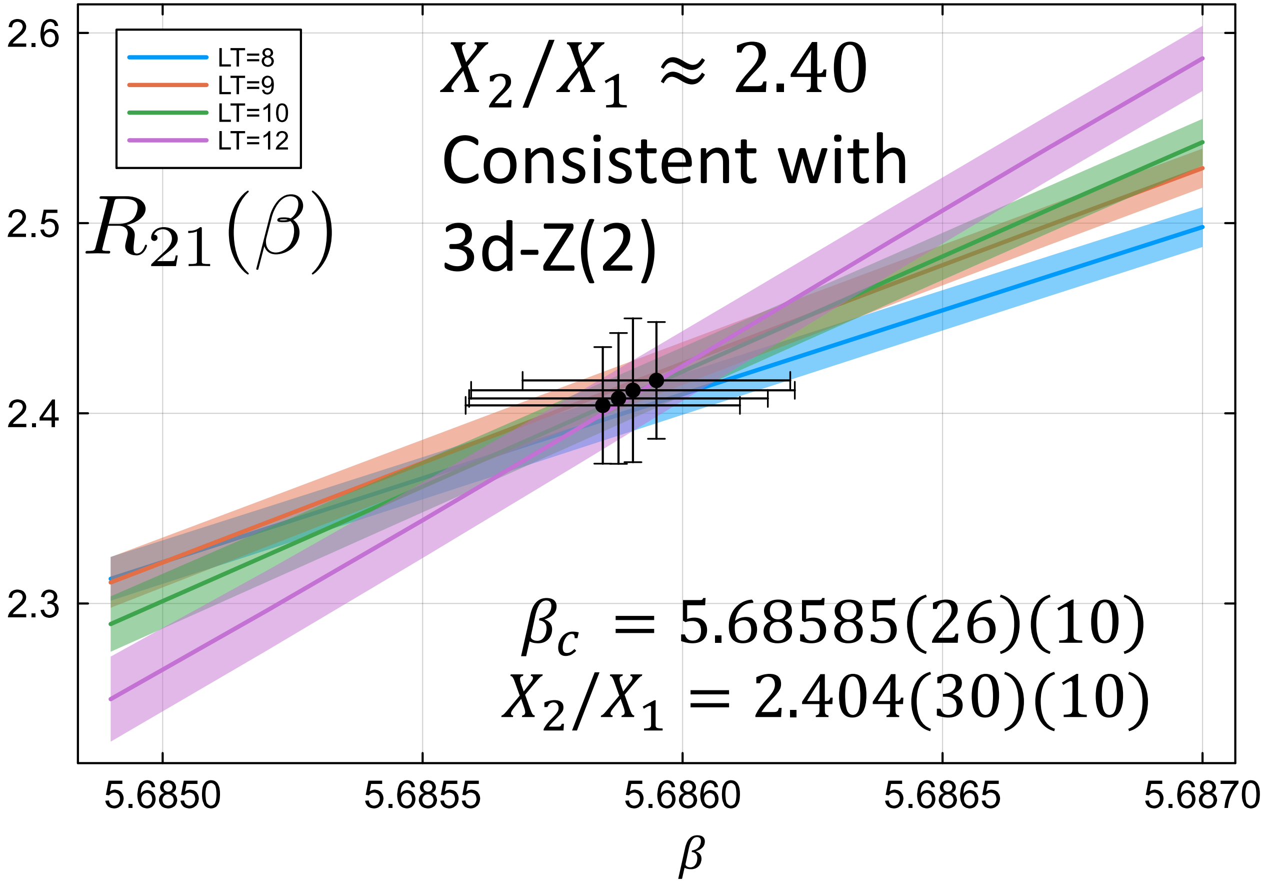


Kiyohara, *et al.*

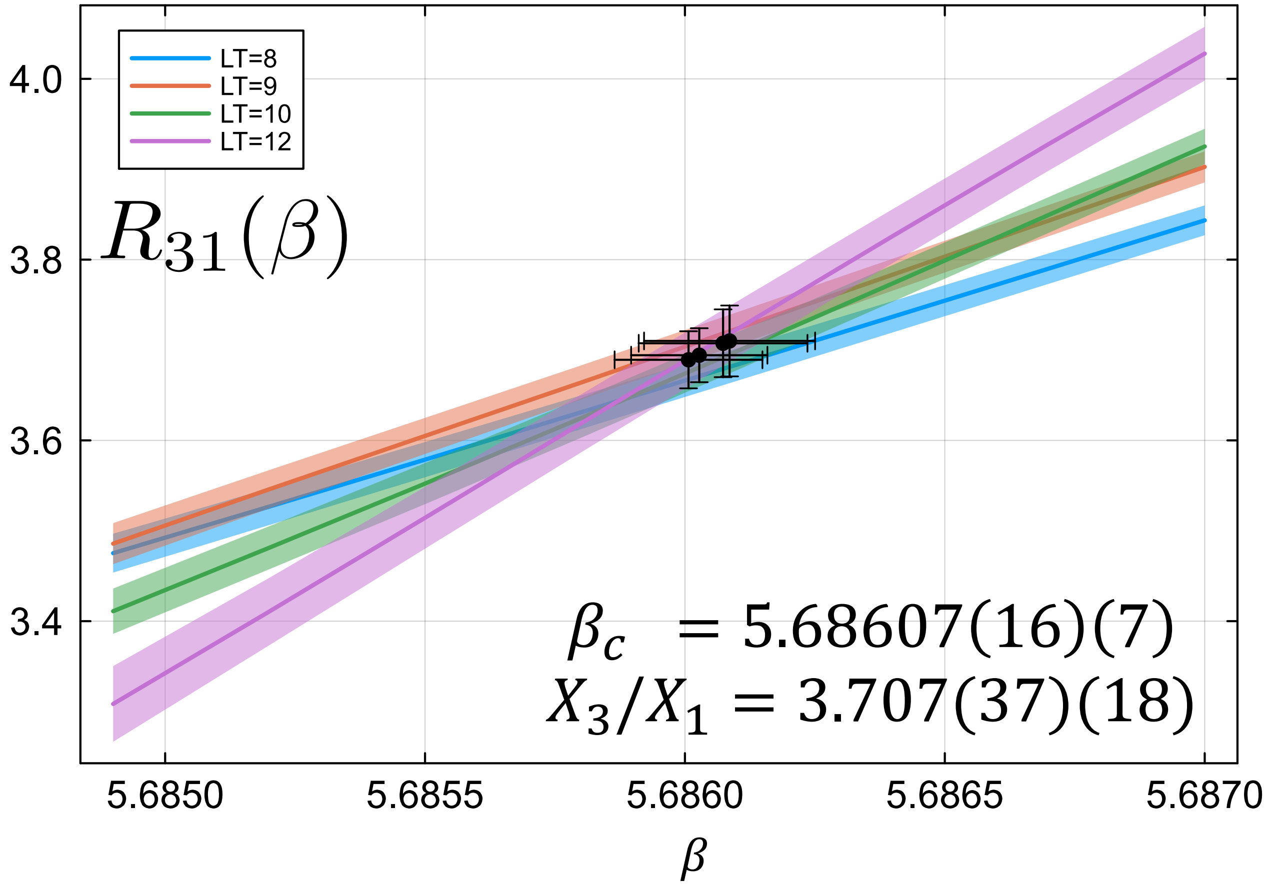


Heavy-Quark QCD

$2^{nd}/1^{st}$ LYZ Ratio



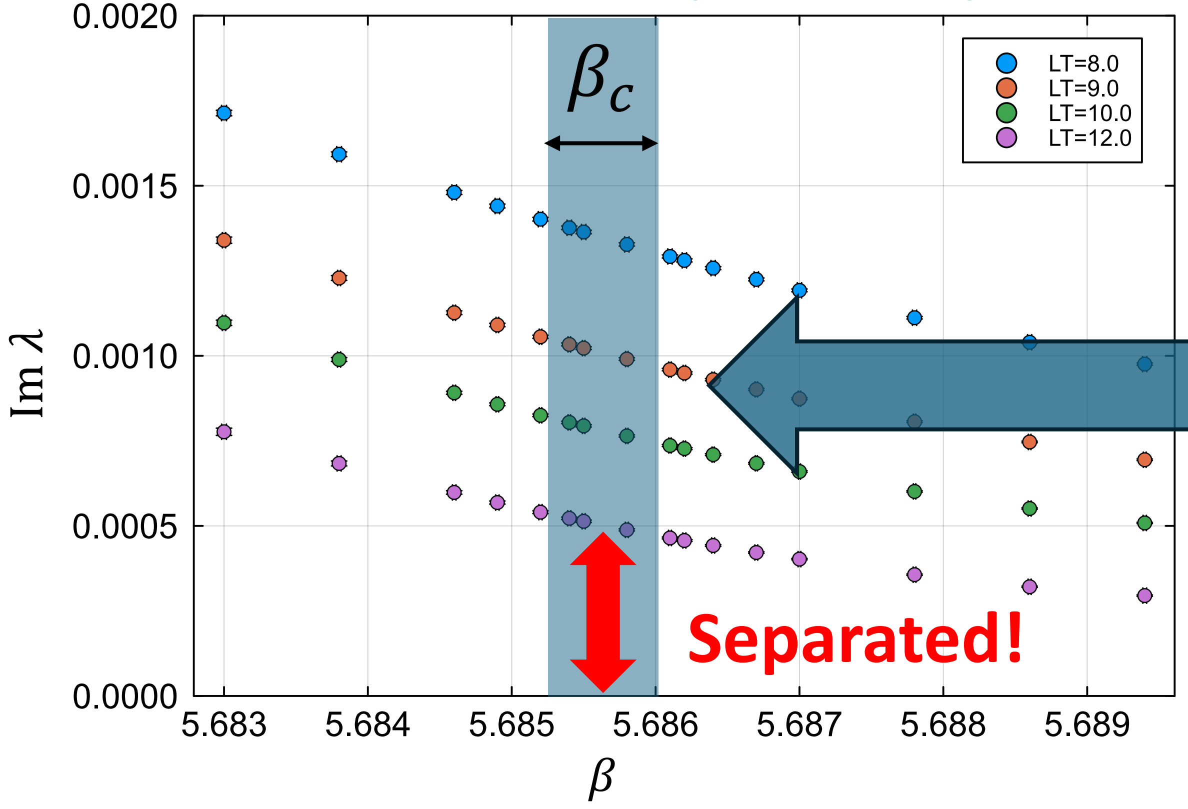
$3^{rd}/1^{st}$ LYZ Ratio



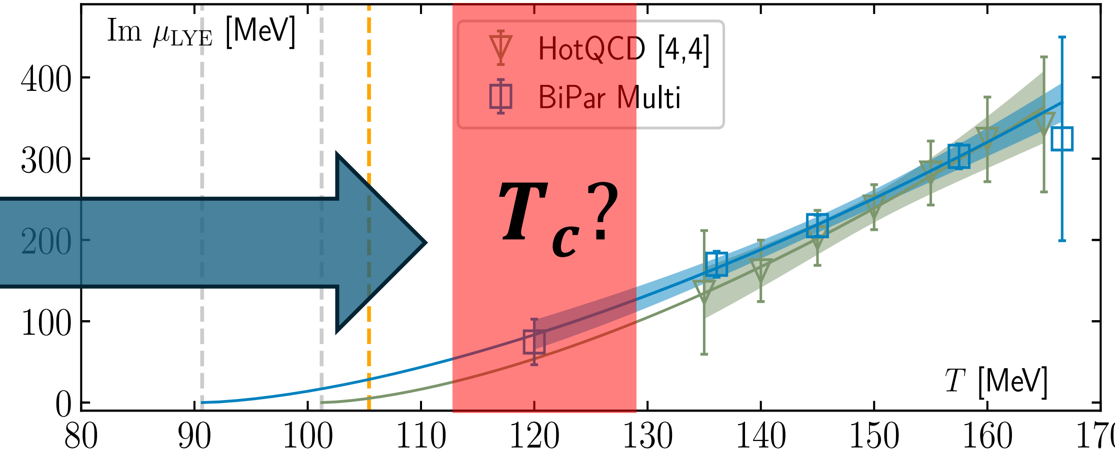
- Consistent with $\beta_c = 5.68578(22)$ from Binder cumulant (Kiyohara et al.)
- 3rd/1st ratio gives more precise result

Speculation to Full-QCD result

1st LYZ data (HQ-QCD)



$N_f = 2 + 1$ (Clarke, *et al.*)



- Finite-size effects is important. LYZ is separated from Re-axis at the CP.
- Once the 2nd LYZ is obtained, the LYZ ratio method is applicable!

Summary

- ✓ We studied the finite-size scaling (FSS) of the Lee-Yang zeros (LYZ) around general critical points (CP).
- ✓ We proposed the **LYZ ratio method**, which is a novel method to determine the location of a CP on the phase diagram based on the ratio of LYZ.
- ✓ Mixing effect in LYZ Ratio is suppressed more strongly than Binder cumulant.
- ✓ The LYZ ratio method is applied to numerical analyses in the 3d Potts model and the heavy-quark QCD.

Future work

- ☐ Roberge-Weiss phase transition
- ☐ QCD critical point at nonzero μ
- ☐ Need the info. of 2nd LYZ