Lattice EFT Test Finite-Volume Formalism for Two-Hadron Matrix Elements

Raul Briceño, Andrew Jackura,

Charles Kacir, Joseph Moscoso, Amy Nicholson

THE UNIVERSITY *of* NORTH CAROLINA *at* CHAPEL HILL

LATTICE 2024

LIVERPOOL

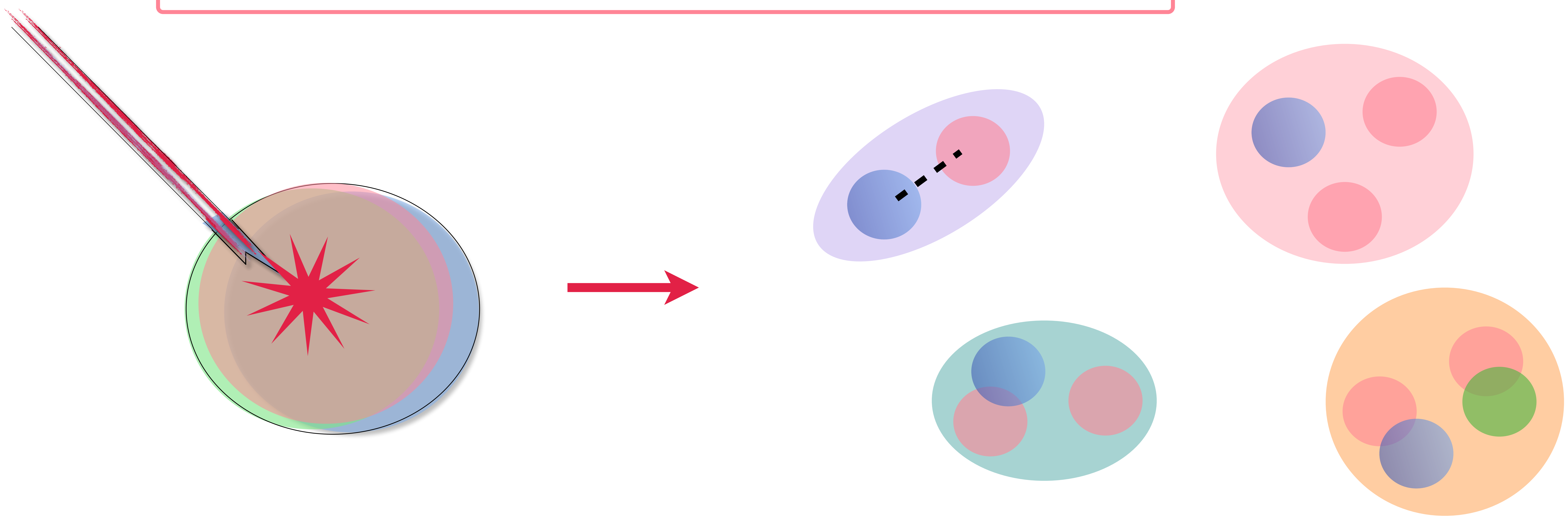
NSF Graduate Research Fellowship Program

Electroweak Hadrons

- **Hadron Spectroscopy** and internal structure of QCD states

Tuesday (4 PM) Hadronic and Nuclear Spectrum and Interactions

The timeline pion form factor and other applications of $I = 1$ $\pi\pi$ scattering
Nolan Miller
Timelike pseudoscalar form factors in a coupled channel from LQCD
Felipe Ortega-Gama

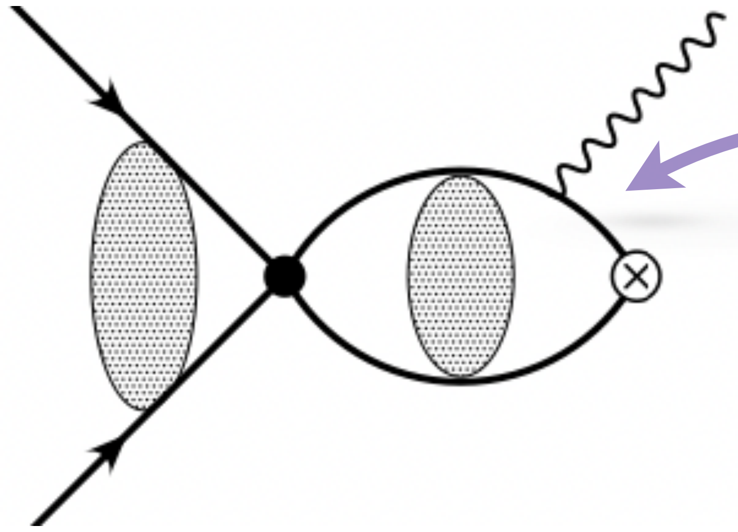


Electroweak Hadrons

- **Hadron Spectroscopy** and internal structure of QCD unstable states

Neutrino-nuclear reactions are critical!

- **Proton-Proton fusion, νd scattering**
- **Double β-decay Transitions**



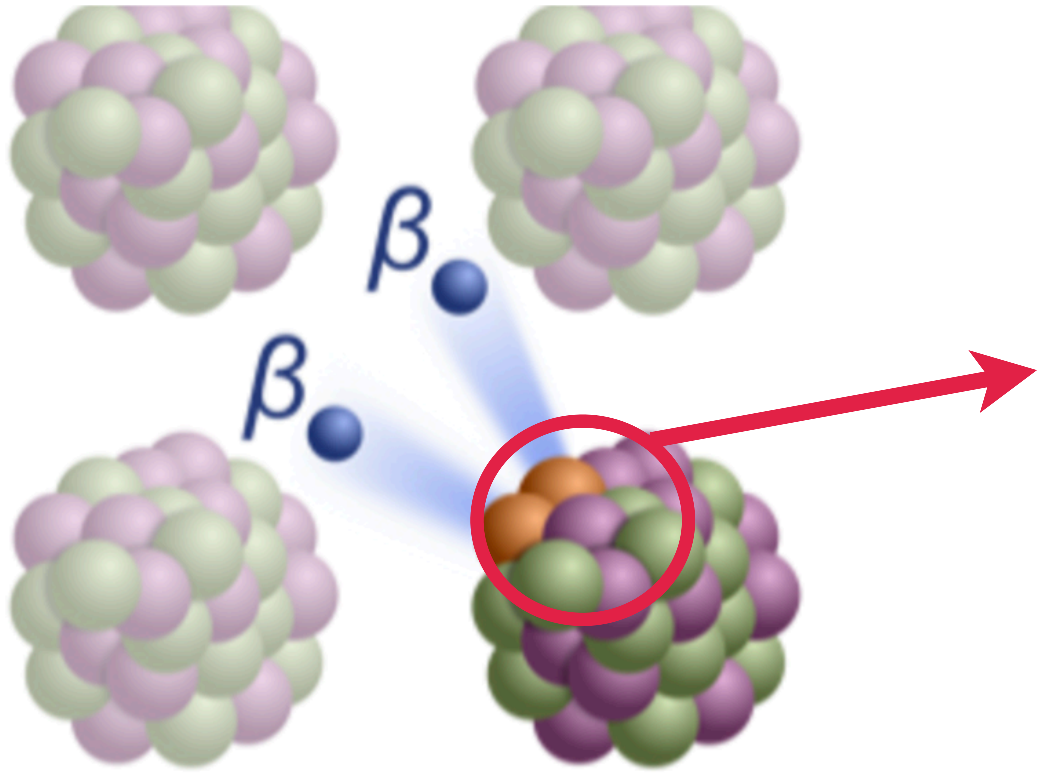
Single-weak current

Wednesday(4 PM). *Zi-Yu Wang*
Lattice calculation
Proton-Proton Fusion Matrix Element

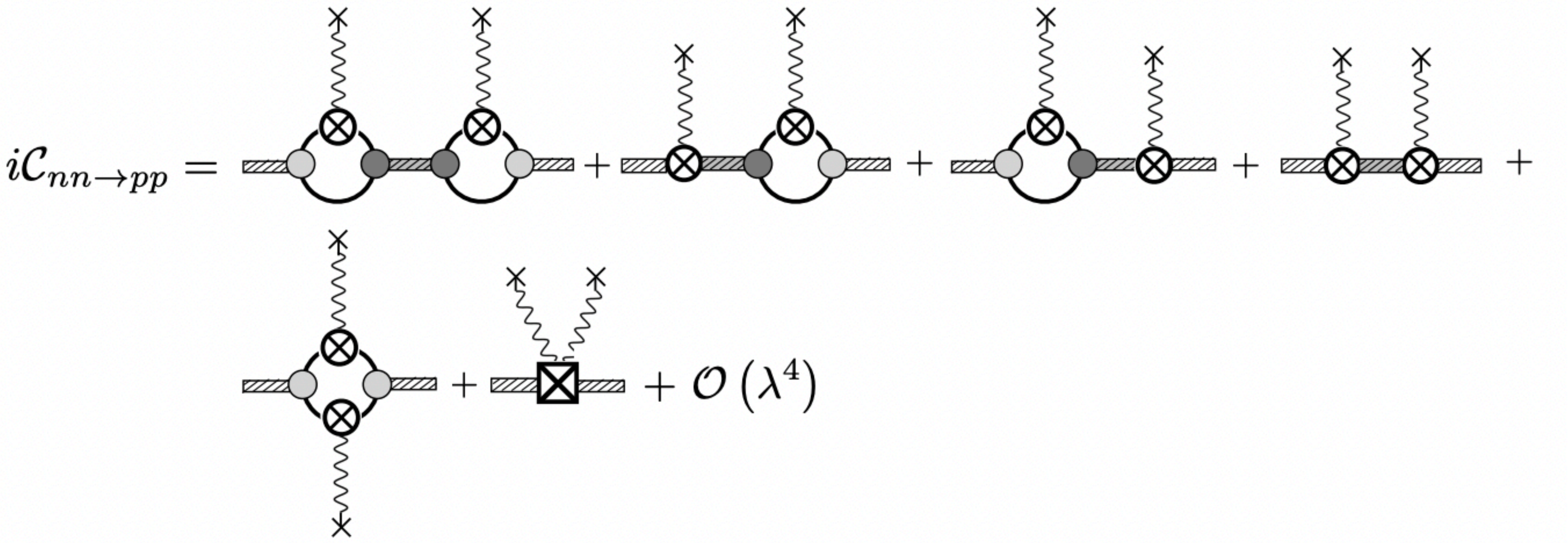
$2\nu\beta\beta$

L-violating
$0\nu\beta\beta$

LEGEND



Low-energy $NN \to NN$



[Tiburzi, Wagman, Winter, et.al. (2017);1702.02929]

Lattice QCD

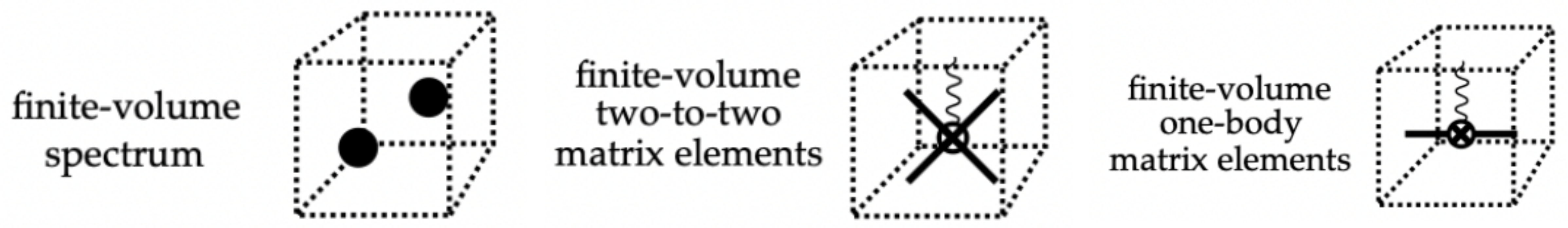
- **LQCD calculations of multi-hadron systems are maturing to include** **Meson-Baryon systems**, **Coupled-Channel Systems**, **Meson Form Factors**

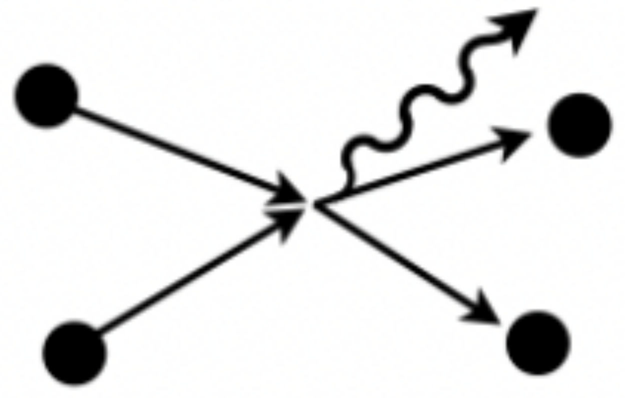
- Lattice QCD for $0\nu\beta\beta$ [Ciriliano, Detmold, Nicholson, Shanahan (2020);2003.08493]

  - **Connect to EFTs with transition amplitudes (Matrix Elements)**

    NN
    - Long-Distance Nuclear Matrix Elements [Davoudi, Detmold, Fu et.al. (2024);2402.09362]
    - Heavy Physics (Short-Range) Contributions [Nicholson, Berkowitz, Monge-Camacho et.al. (2018);1805.02634]

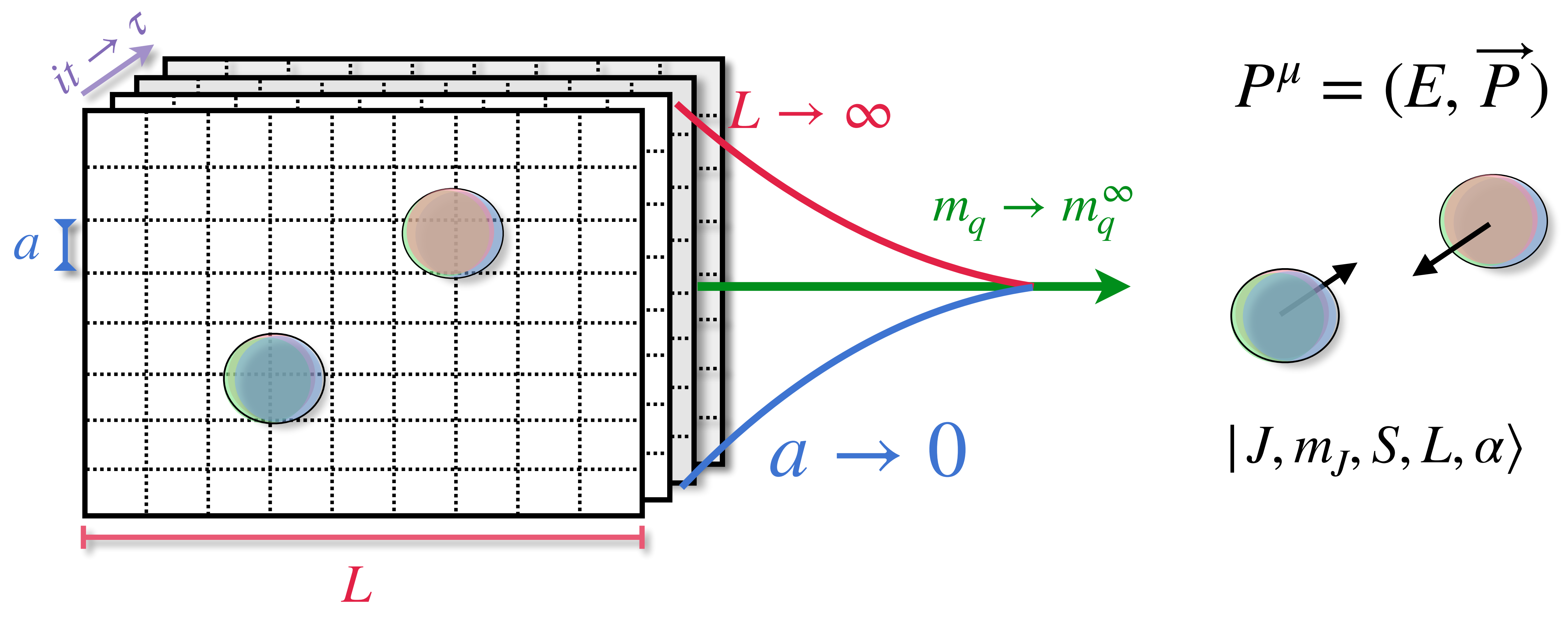
FV Formalism required for LQCD calculations

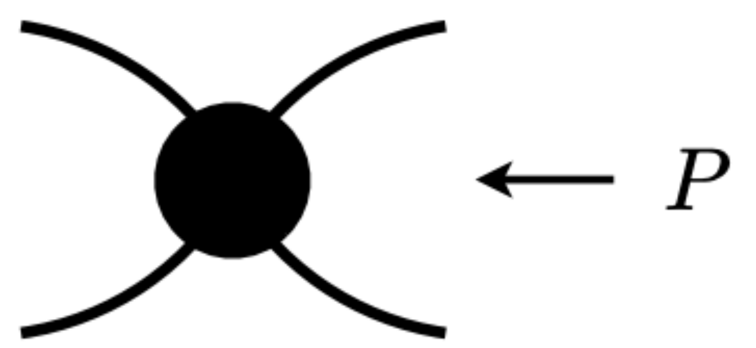




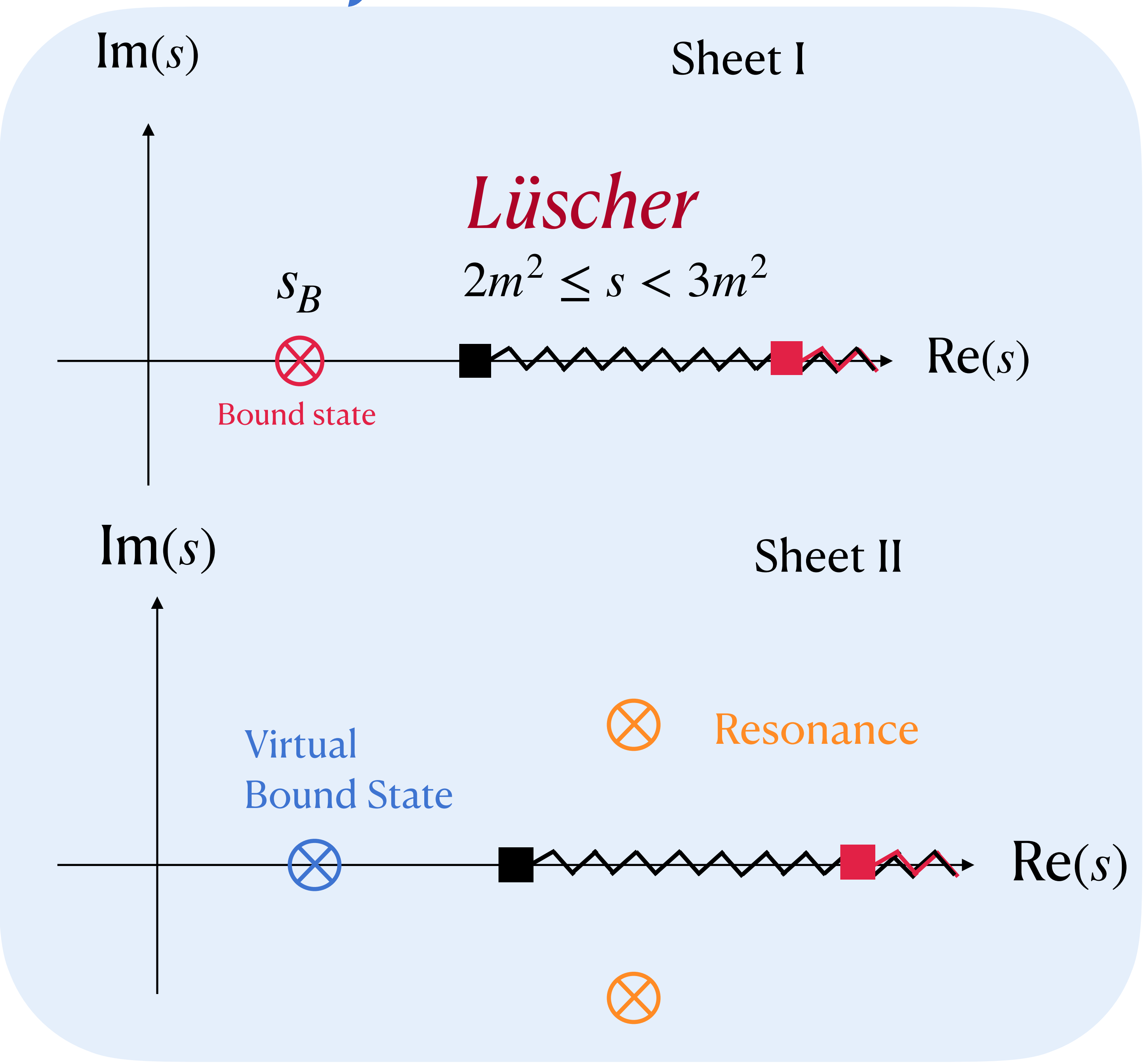
Background

LATTICE → Reality   2 → 2 *Elastic Scattering*

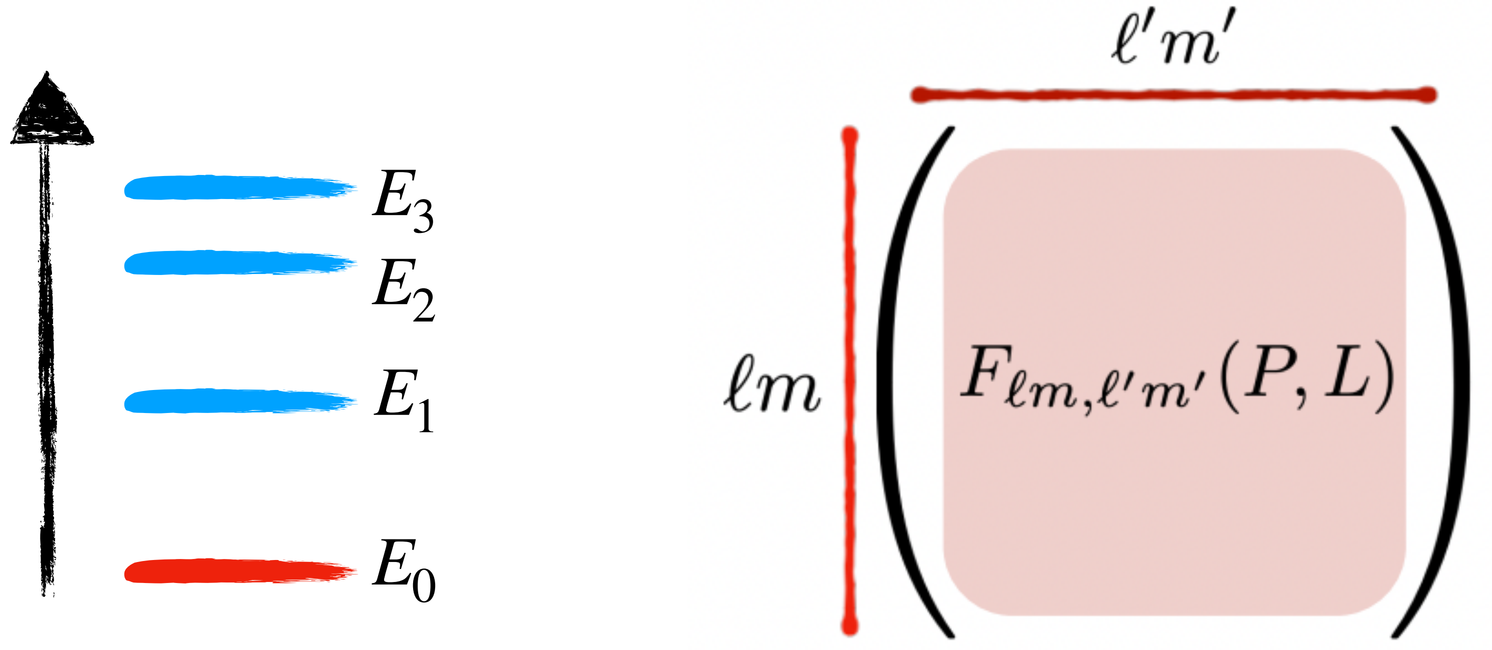


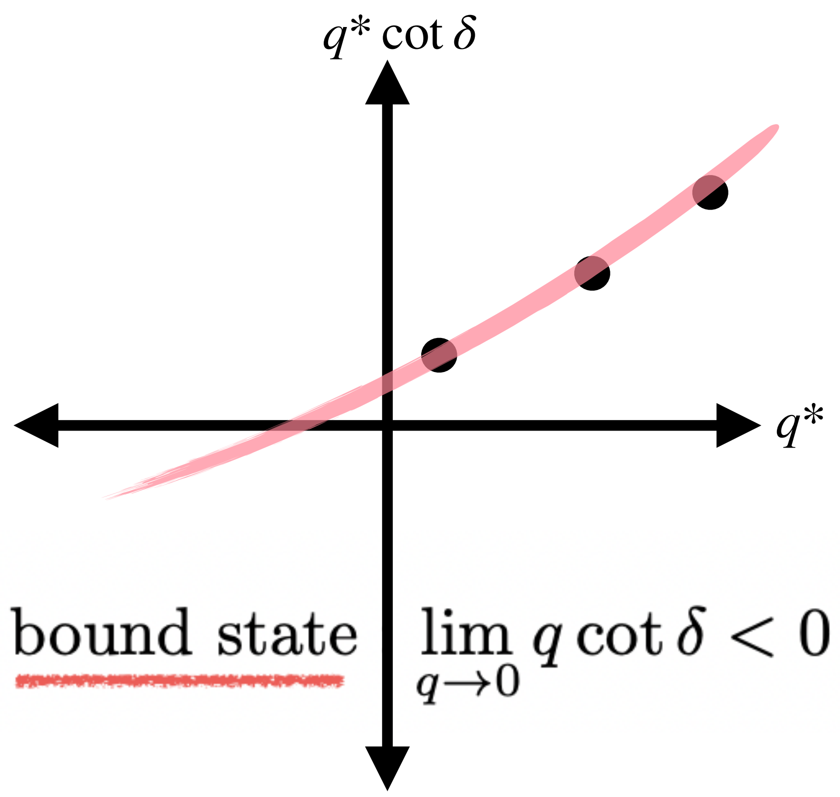


Analytic Structure



◆ **Lüscher Formalism** $\det\left(F^{-1}(E_n, L) + \mathcal{M}(E_n)\right) = 0$

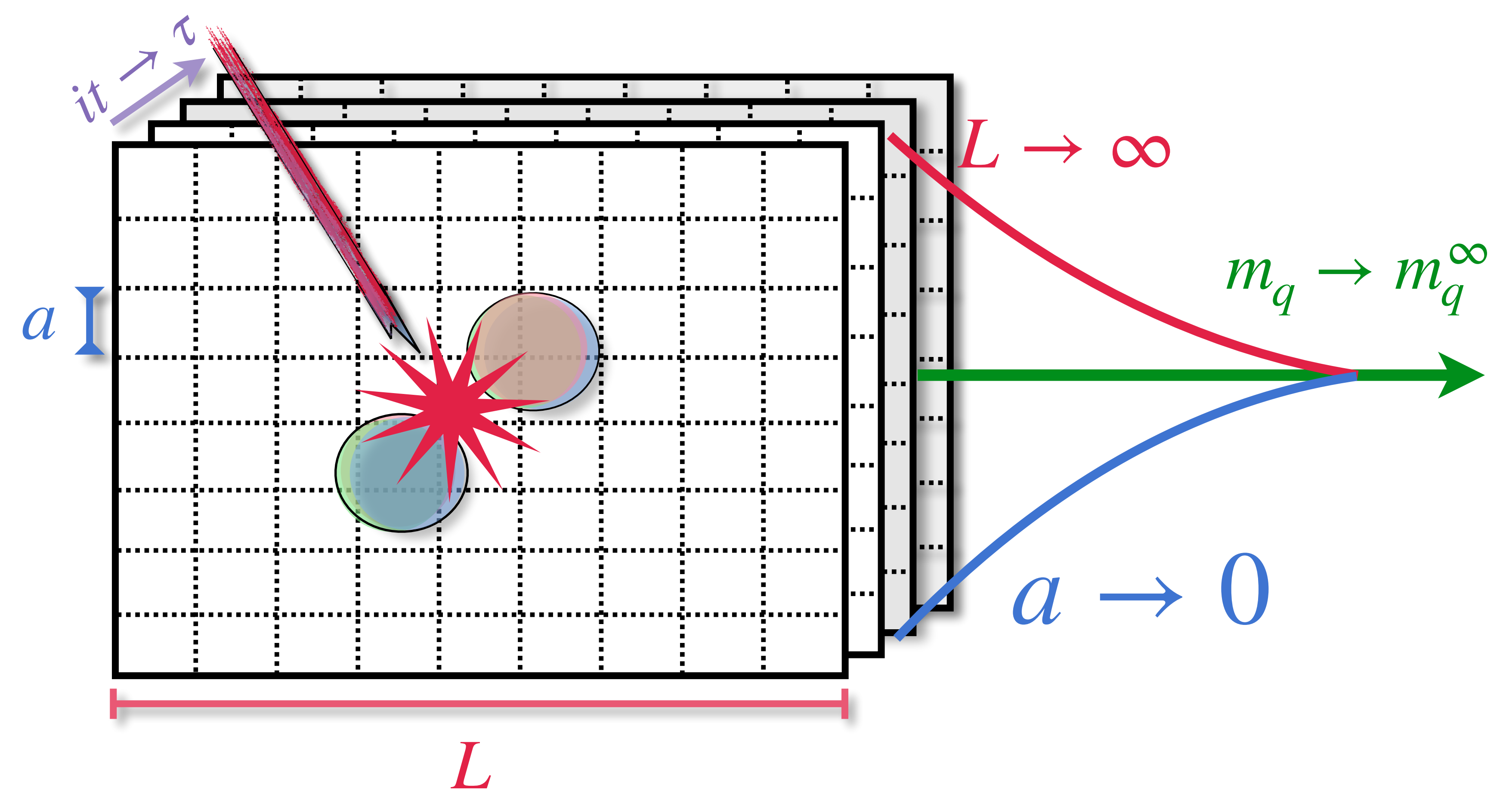


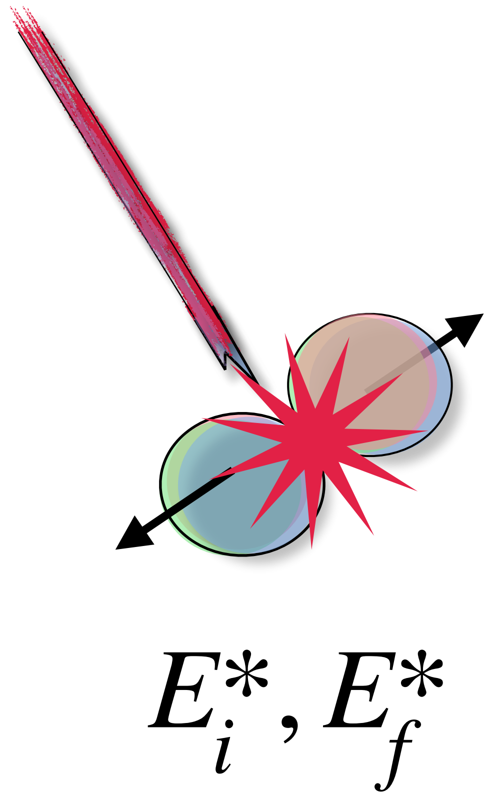


LATTICE 2024

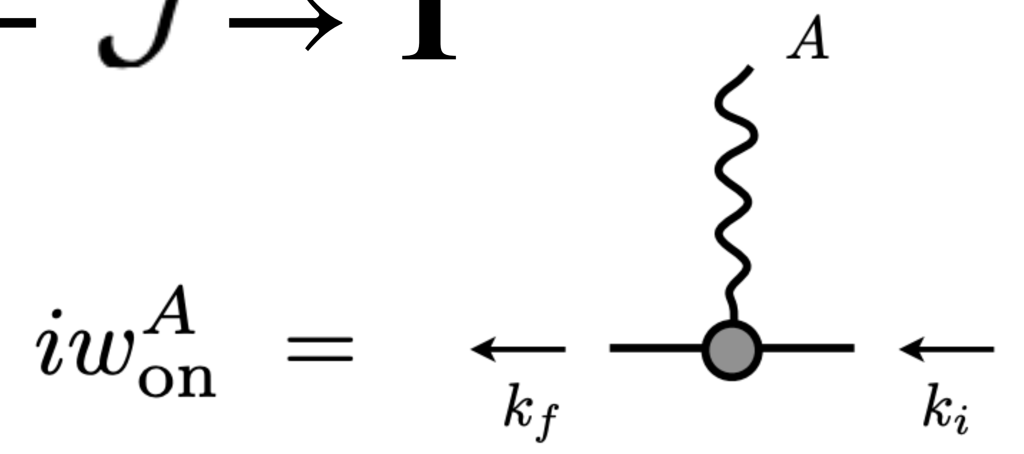
LATTICE → Reality

$\mathbf{n} + \mathcal{J} \rightarrow \mathbf{m}$ *Transition Amplitude*

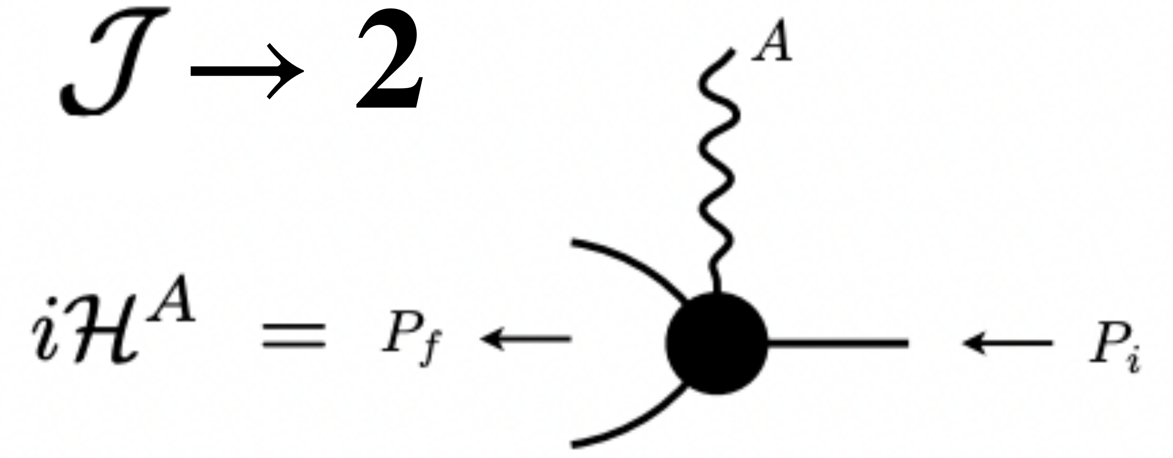




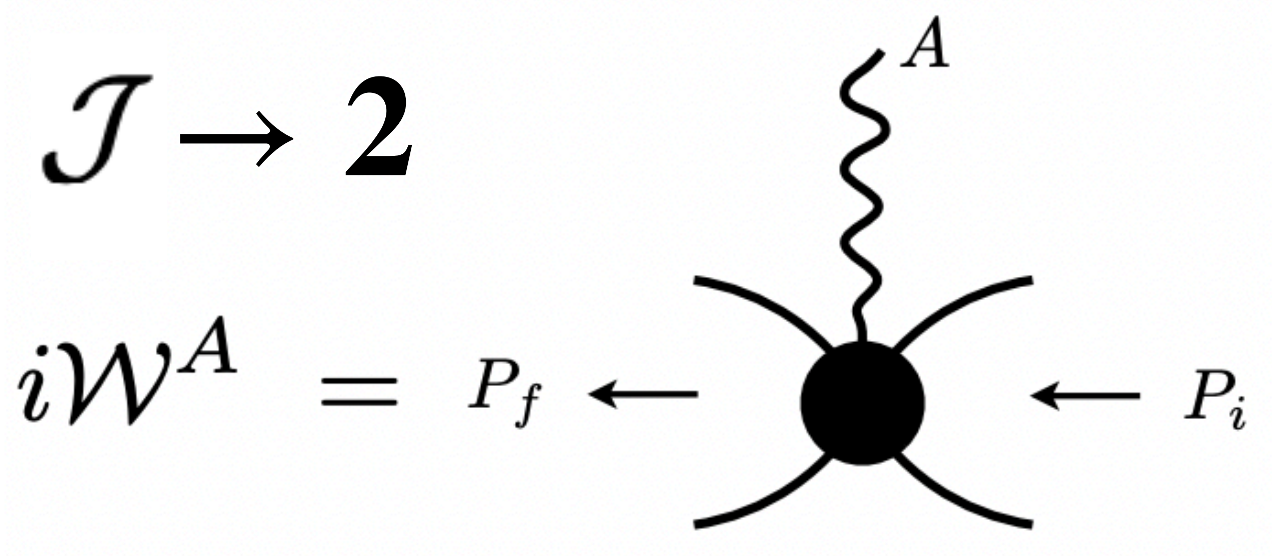
$\mathbf{1} + \mathcal{J} \rightarrow \mathbf{1}$



$\mathbf{1} + \mathcal{J} \rightarrow \mathbf{2}$



$\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$



- ***Lellouch-Lüscher (LL)*** $\gamma^*\pi \rightarrow \pi\pi$ ρ

  [Lellouch, Lüscher (2001); 0003023] [Briceño, Dudek,Edwards et. al.(2016); 1604.03530]

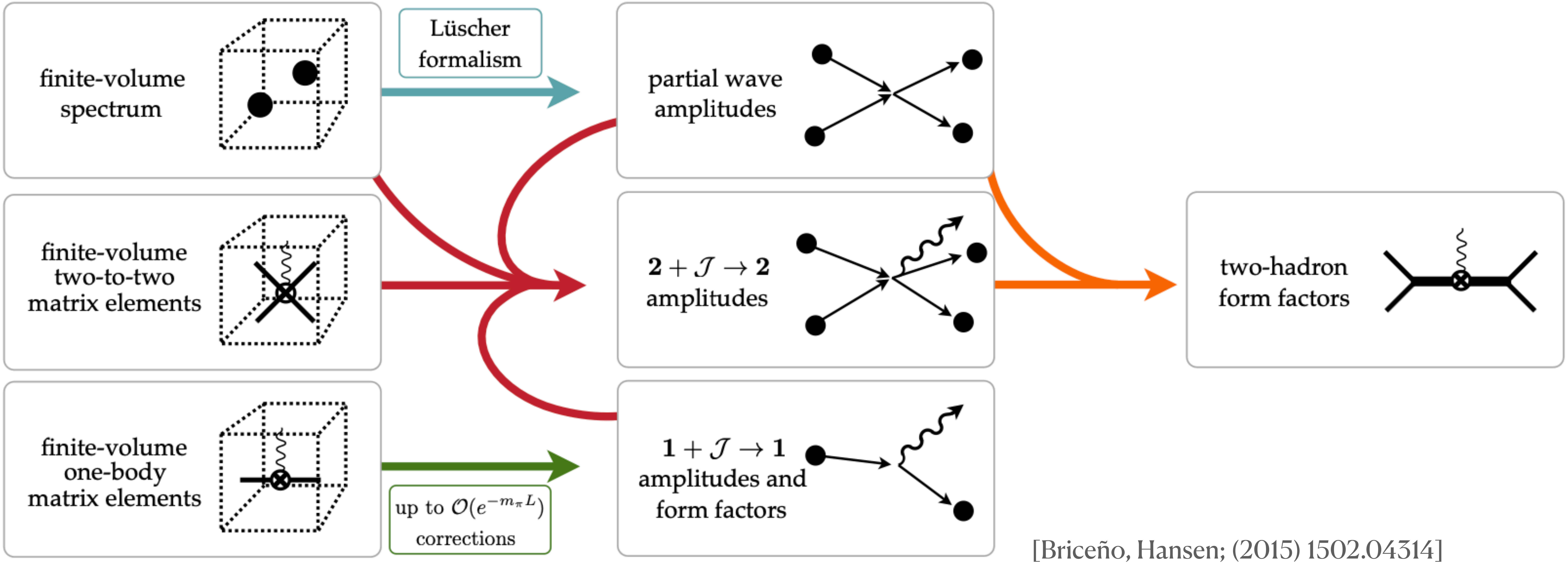
  $K \rightarrow \pi\pi$ $\langle \pi\pi | \mathcal{L}_W | K \rangle$

- ***Recently developed Formalism*** →

  Covariant, Relativistic, New Analytic Properties

LATTICE 2024

$\langle 2|\mathcal{J}|2\rangle_L$ Roadmap



[Briceño, Hansen; (2015) 1502.04314]

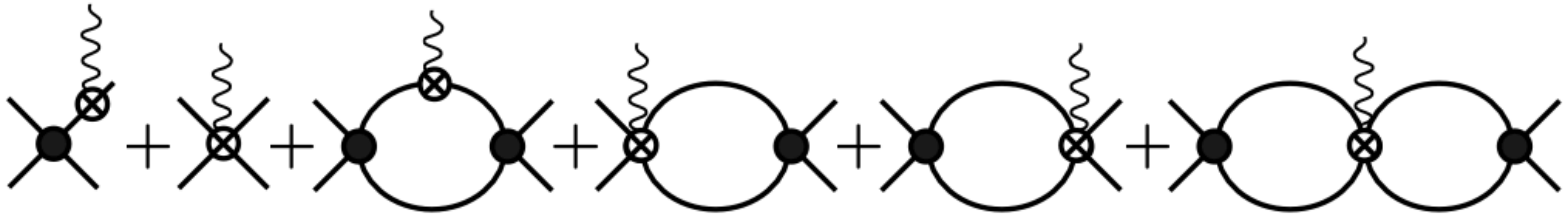
[Baroni, Briceño, Hansen, Ortega-Gama (2019); 1812.10504]

Formalism

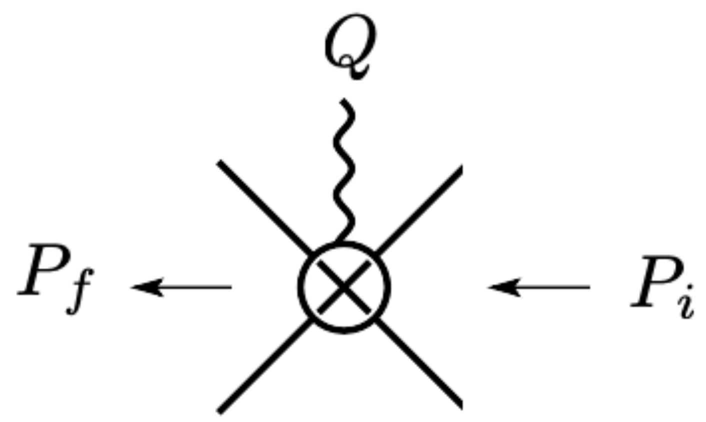
physical $\mathbf{2} + \mathcal{J} \to \mathbf{2}$ matrix element

$$\mathcal{W}_{\mu_1\cdots\mu_n}(P_f, k'; P_i, k) \equiv \langle P_f, k'; \text{out}|\mathcal{J}_{\mu_1\cdots\mu_n}(0)|P_i, k; \text{in}\rangle_{\text{conn.}}$$

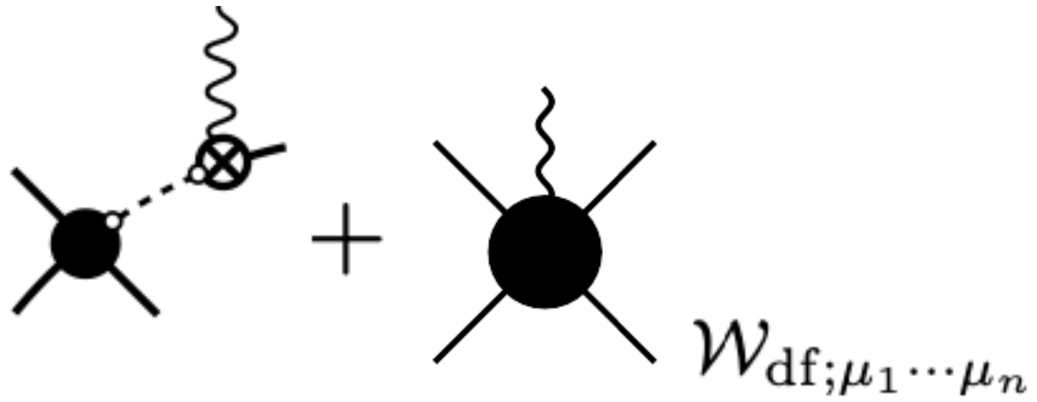
$\mathcal{W}_{\mu_1\cdots\mu_n} =$



$=$



$=$

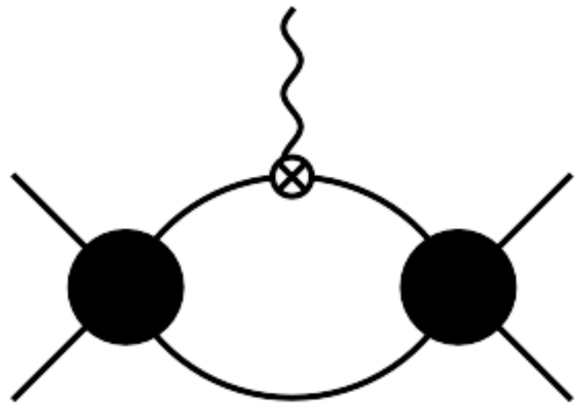


$\longleftrightarrow$ $\langle 2|\mathcal{J}|2\rangle_L$ finite-volume two-to-two matrix elements

Hadron (QCD)

1+J->1 Matrix Element

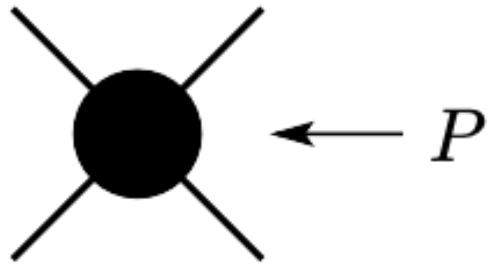
$\mathcal{J}_{\mu_1\cdots\mu_n}(0)$ Local Current



Triangle Diagram

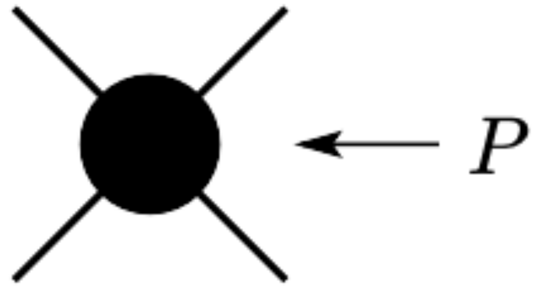
$G_{\mu_1\cdots\mu_n}(P_f, P_i, L)$

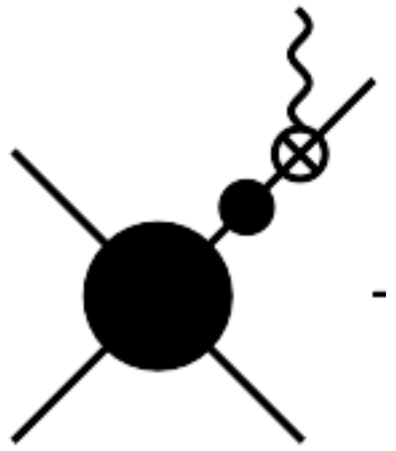
$i\mathcal{M} =$



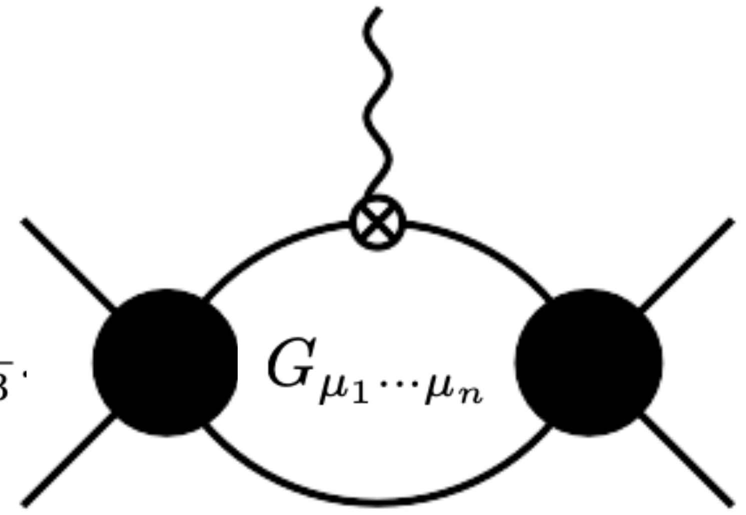
2 -> 2 Scattering Amplitude

FV Analytic Structure

$i\mathcal{M} =$ 



$$\left[\frac{1}{L^3}\sum\!\!\!\!\!\!\int_{\mathbf{k}}\right] \equiv \frac{1}{L^3}\sum_{\mathbf{k}\in(2\pi/L)\mathbb{Z}^3} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}$$

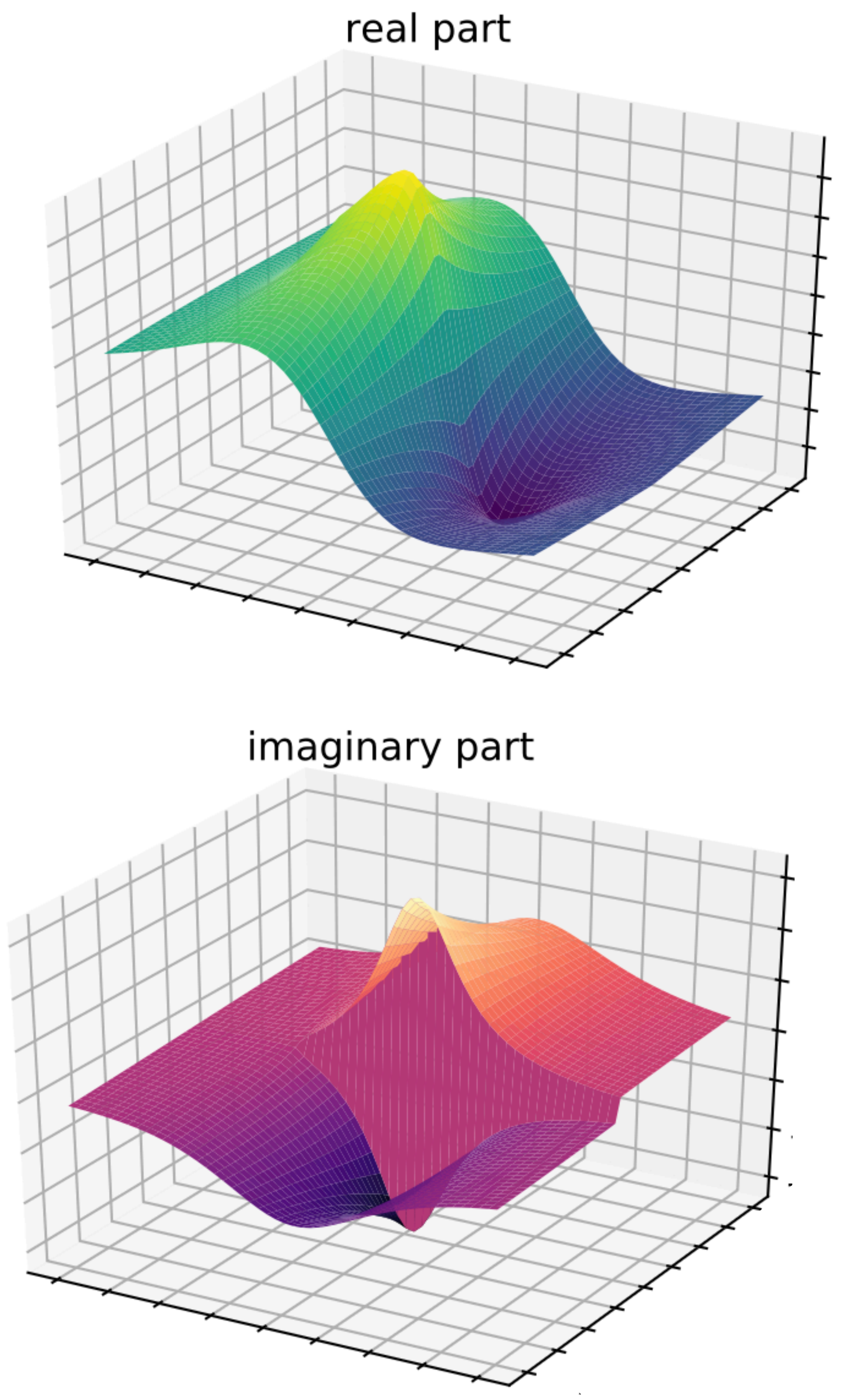


Kinematic Singularities

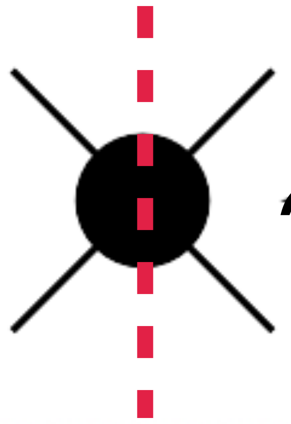
Landau Singularities

On-shell Intermediate States

Isolated Poles

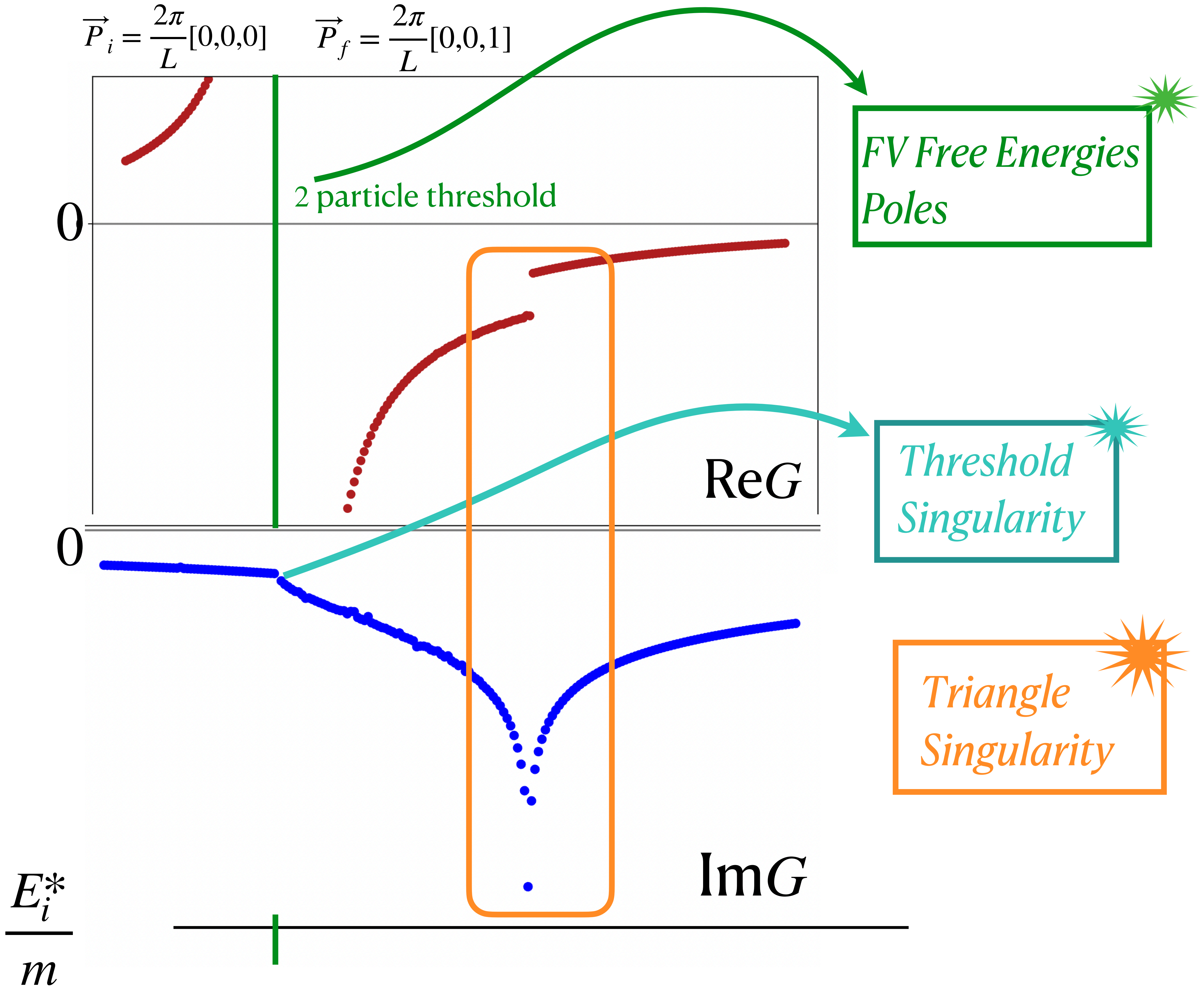


 $\sim f(Q^2)$

$i\mathcal{M} =$  $\sim \frac{\xi\, q^\star}{8\pi\sqrt{s}}$

$$iD_\alpha(k) = \frac{i}{k^2 - m_\alpha^2 + i\epsilon}\,.$$

✱ $k = \frac{2\pi}{L}\vec{n}$



$\langle 2|\mathcal{J}|2\rangle_L$ Consistency Checks

Conserved currents and bound states [Briceño, Hansen, Jackura; 1909.10357]

- Charge Operator $\int d^3\mathbf{x}\,\mathcal{J}^0(x) \longrightarrow L^3\mathcal{J}^0(0) \equiv \widehat{\mathrm{Q}}$

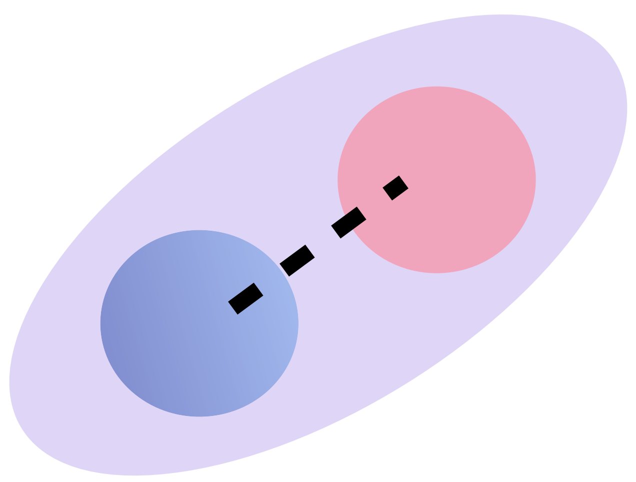
Current Conservation!

$$\langle P_n, L|\,\widehat{\mathrm{Q}}\,|P_n, L\rangle = \mathrm{Q}_0$$

$$\mathrm{Q}_0 = \mathrm{Q}_{\mathrm{p}} + \mathrm{Q}_{\mathrm{n}}$$

@ $Q^2 = -(P_f - P_i)^2 = 0$

- Conserved Vector Current $\partial_\mu \mathcal{J}^\mu(x) = 0$



S-wave **bound-state**

1 $$\mathcal{W}^\mu(P_f, P_i) = (P_i + P_f)^\mu\, F_{\mathrm{B}}(Q^2)\, \frac{i^2(ig)^2}{(s_f - s_{\mathrm{B}})(s_i - s_{\mathrm{B}})}$$

2 $$\langle 2|\mathcal{J}|2\rangle_L^B \longrightarrow \langle 2|\mathcal{J}|2\rangle^B + \mathcal{O}(e^{-\kappa_{\mathrm{B}} L})$$

Binding Momentum

$\langle 2|\mathcal{J}|2\rangle_L$ Consistency Checks

Conserved currents and bound states [Briceño, Hansen, Jackura (2019); 1909.10357]

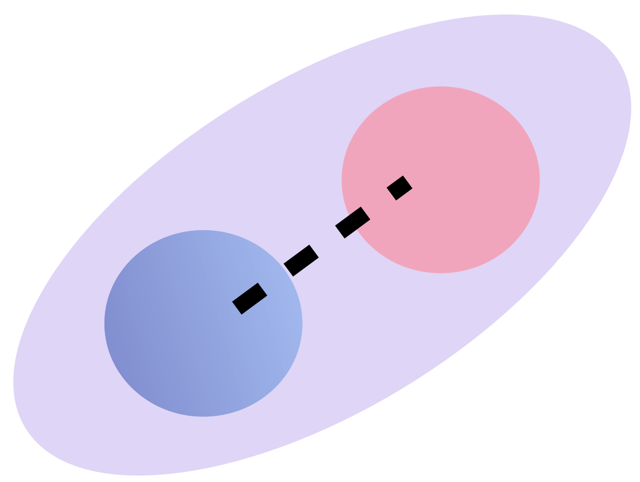
- Charge Operator $\int \mathrm{d}^3\mathbf{x}\,\mathcal{J}^0(x) \longrightarrow L^3\mathcal{J}^0(0) \equiv \widehat{\mathrm{Q}}$
- Conserved Vector Current $\partial_\mu \mathcal{J}^\mu(x) = 0$

Current Conservation!

$$\langle P_n, L|\,\widehat{\mathrm{Q}}\,|P_n, L\rangle = \mathrm{Q}_0$$

$$\mathrm{Q}_0 = \mathrm{Q}_p + \mathrm{Q}_n$$

$$@ \quad Q^2 = -(P_f - P_i)^2 = 0$$

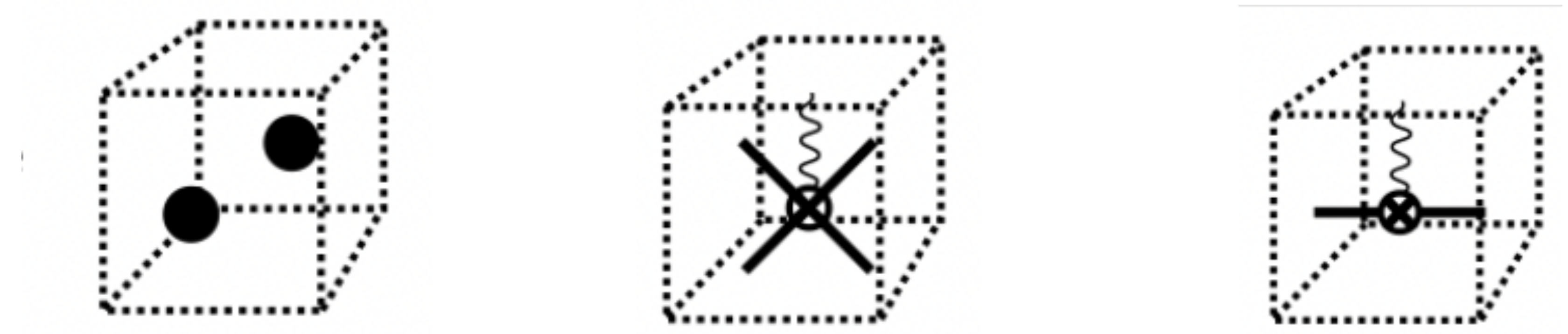


S-wave **bound-state**

Test using numerical EFT

Single External Vector Current

[Briceño, Jackura;, Ortega-Gama, Sherman (2020) 2012.13338]



Lattice Pionless EFT

Endres, Kaplan, Lee, Nicholson (2011)

- **Low energy Nuclear EFT**
  2 point-like non-relativistic nucleons interacting via delta functions

Energies << m_π (same condition as Lüscher)

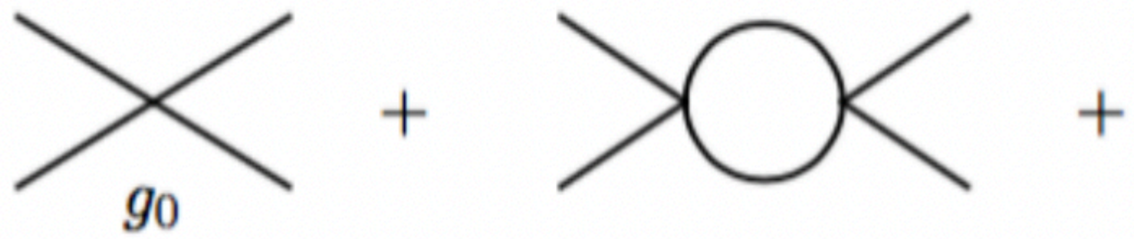
LECs

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \left(i\partial_\tau + \frac{\nabla^2}{2M} \right) \psi + g_0 \left(\psi^\dagger \psi \right)^2 + \frac{g_2}{8} \left[(\psi\psi)^\dagger \left(\psi \overleftrightarrow{\nabla}^2 \psi \right) + \text{h.c.} \right] + \cdots$$

Nucleon Mass

Tune coupling by using s-wave scattering amplitude

$$A = \frac{4\pi}{M} \frac{1}{p\cot\delta - ip} =$$



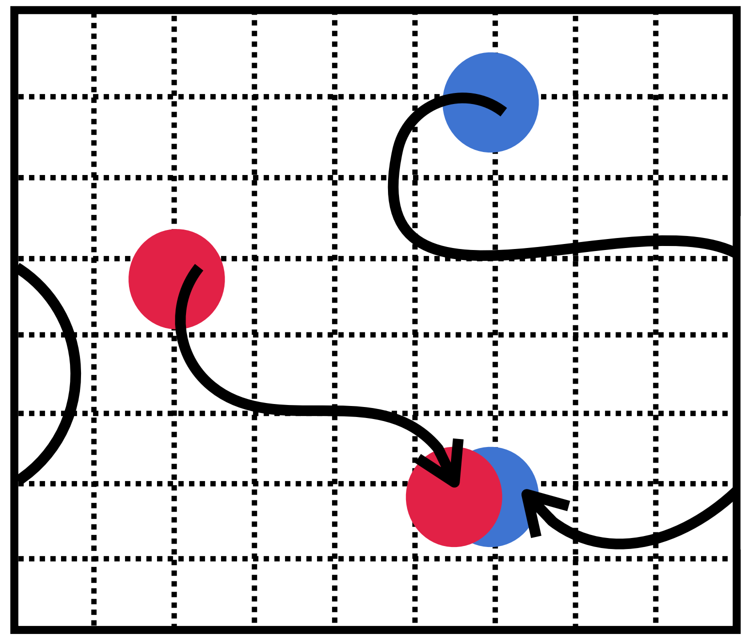
ERE $p\cot\delta = -\frac{1}{a} + \frac{1}{2} r_0 p^2 + r_1 p^4 + \cdots ,$

$$A = g_0 \left[1 + \sum_n (g_0 I_0)^n \right] = \frac{g_0}{1 - g_0 I_0} .$$

Endres, Kaplan, Lee, Nicholson (2011)

Discretize

$$\mathcal{L}_{\text{eff}} = \psi^\dagger \left(i\partial_\tau + \frac{\nabla^2}{2M} \right) \psi + g_0 \left(\psi^\dagger \psi\right)^2 + \frac{g_2}{8} \left[(\psi\psi)^\dagger \left(\psi \overleftrightarrow{\nabla}^2 \psi\right) + \text{h.c.} \right] + \cdots$$



$L^3 \times N_\tau$

$\psi = (\psi^\uparrow, \psi^\downarrow)$

PBC

$$\partial^{(L)}_{\hat{k}} f_j = \frac{1}{b_s} \left[f_{j+\hat{k}} - f_j \right]$$

$$\partial^{(L)}_{\tau} \psi_{\vec{x},\tau} = \frac{1}{b_\tau} \left[\psi_{\vec{n},\tau} - \psi_{\vec{n},\tau-1} \right]$$

$$\nabla^2_L f_j = \sum_k \frac{1}{b_s^2} \left[f_{j+\hat{k}} + f_{j-\hat{k}} - 2 f_j \right]$$

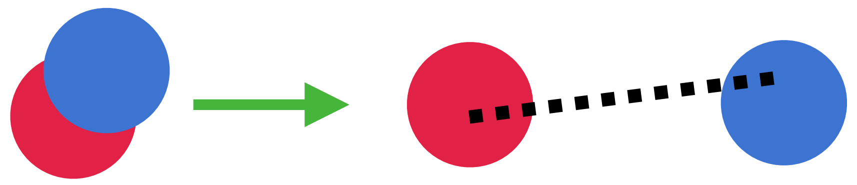
Enforcing continuum limit

$$S_{\text{KE}} \propto \sum_j \psi_j^\dagger \nabla_L^2 \psi_j$$

$$S_{\text{free}} = \sum_{\tau,\tau'} \frac{1}{b_\tau} \psi^\dagger_{\tau'} \left[K_0\right]_{\tau,\tau'} \psi_\tau$$

$$\mathcal{L}_{\text{int}} = \sum_{\mathbf{x}} \sqrt{b_\tau g_0}\, \phi_{\mathbf{x},\tau} \psi^\dagger_{\mathbf{x},\tau} \psi_{\mathbf{x},\tau-1}$$

Hubbard-Stratonovich



$Z_2 = \pm 1$

$$D \equiv 1 - \frac{b_s^2 \nabla_L^2}{2}$$

$$X(\phi_\tau) \equiv 1 - \sqrt{g_0}\,\phi_\tau$$

$$S = \frac{1}{b_\tau} \sum_{\tau,\tau'} \psi^\dagger_{\tau'} \left[K(\phi)\right]_{\tau'\tau} \psi_\tau$$

$$K[\phi, N_\tau] \equiv \begin{pmatrix} D & -X(\phi_{N_\tau-1}) & 0 & 0 & . & . & . \\ 0 & D & -X(\phi_{N_\tau-2}) & 0 & . & . & . \\ . & . & . & . & & & \\ . & . & . & & . & & \\ . & . & . & & & D & X(\phi_0) \\ X(\phi_{N_\tau}) & . & . & & & 0 & D \end{pmatrix}$$

APBC

$T = 0$

Endres, Kaplan, Lee, Nicholson (2011)

Transfer Matrix Method (Euclidean Time)

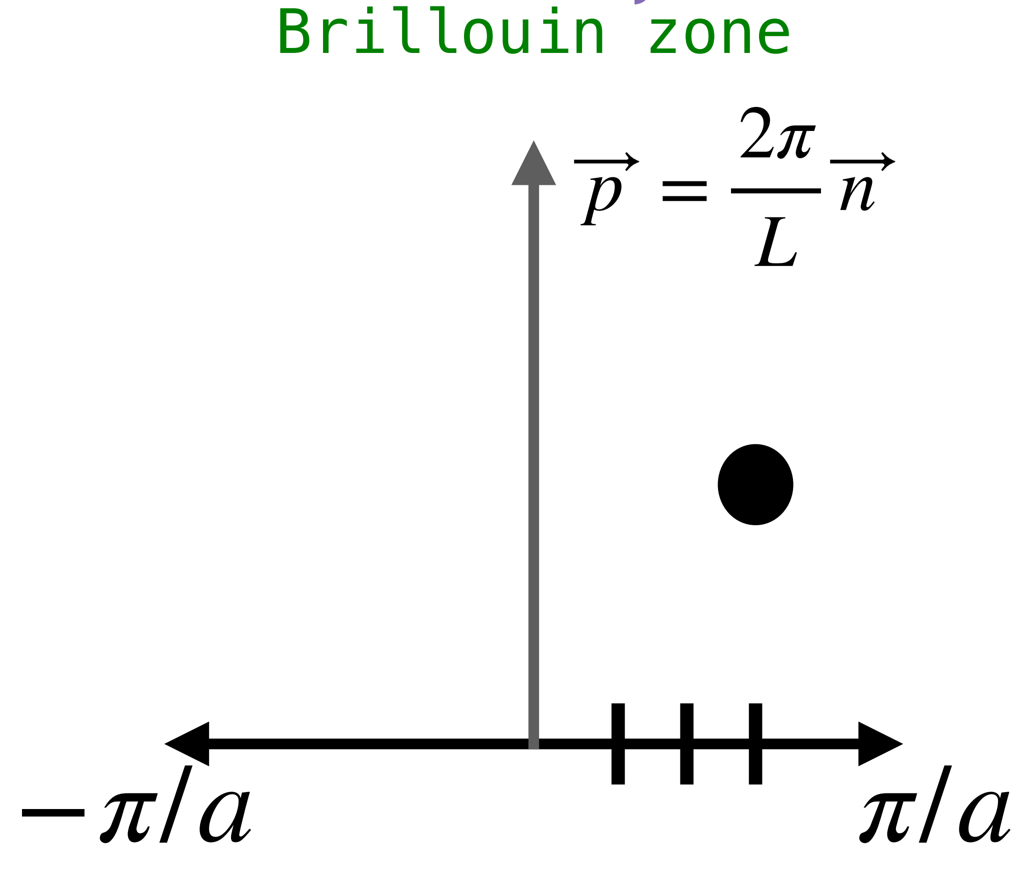
$$C_2(\tau) = \frac{1}{Z}\int \mathcal{D}\phi\mathcal{D}\psi^\dagger\mathcal{D}\psi e^{-S[\psi^\dagger,\psi,\phi]}\Psi^\dagger_{\mathrm{src},2}\Psi_{\mathrm{snk},2} \quad\longrightarrow\quad C(\tau) = \langle \Psi_{\mathrm{snk},2}|e^{-H\tau}|\Psi_{\mathrm{src},2}\rangle$$

$$\longrightarrow \quad \boxed{\mathcal{T} = e^{-H}}$$

$$\langle pq|\mathcal{T}|p'q'\rangle = \frac{\delta_{pp'}\delta_{qq'} + \frac{g_0}{V}\delta_{p+q,p'+q'}}{\sqrt{\xi(p)\xi(q)\xi(q')\xi(p')}} \qquad \xi(p) \equiv 1 + \frac{\Delta(q)}{M}$$

$$p = \frac{2\pi}{L}n$$

Periodic Boundary Conditions

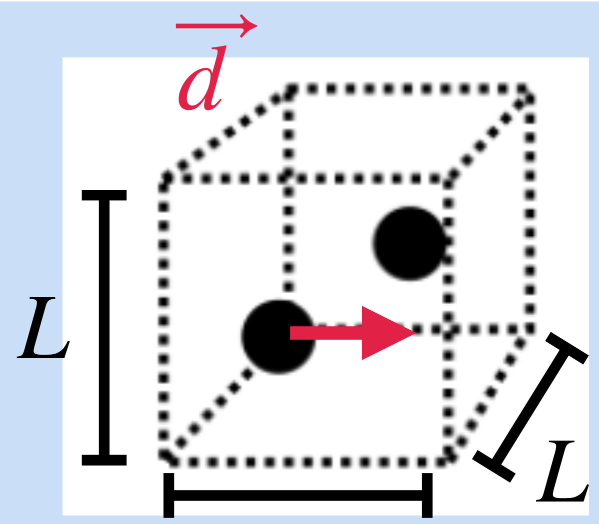


Parameters

L	Lattice size
M	Mass
c	Coupling
$\vec{d}$	Momentum

$$L^3 \times L^3$$

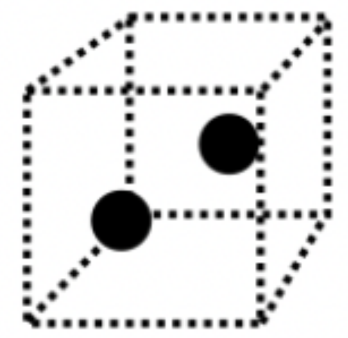
$$\hat{T} = \hat{H}_0 + c\frac{4\pi}{ML^3}\hat{V}_o$$



$$\det\left(T - \lambda\right) = 0$$

$$E_{NR} = \frac{L}{}- \log\lambda_n$$

Preliminary Results

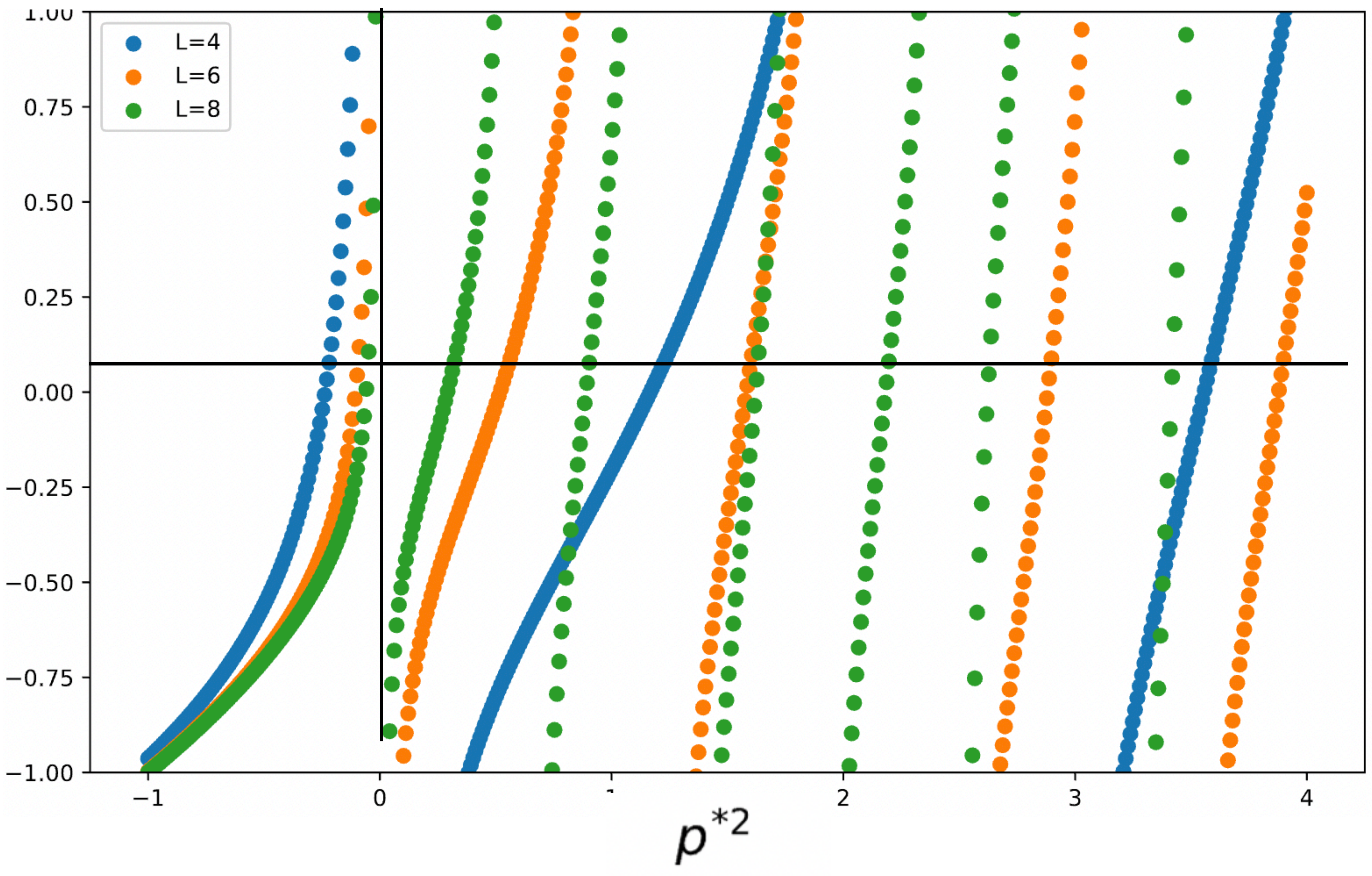


S-wave Non-relativistic States

$$\det\left(F^{-1}(E_n, L) + \mathcal{M}(E_n)\right) = 0$$

$$F(P, L) \sim \ \mathcal{Z}_{00}\left[1; (p_n L/2\pi)^2\right]$$

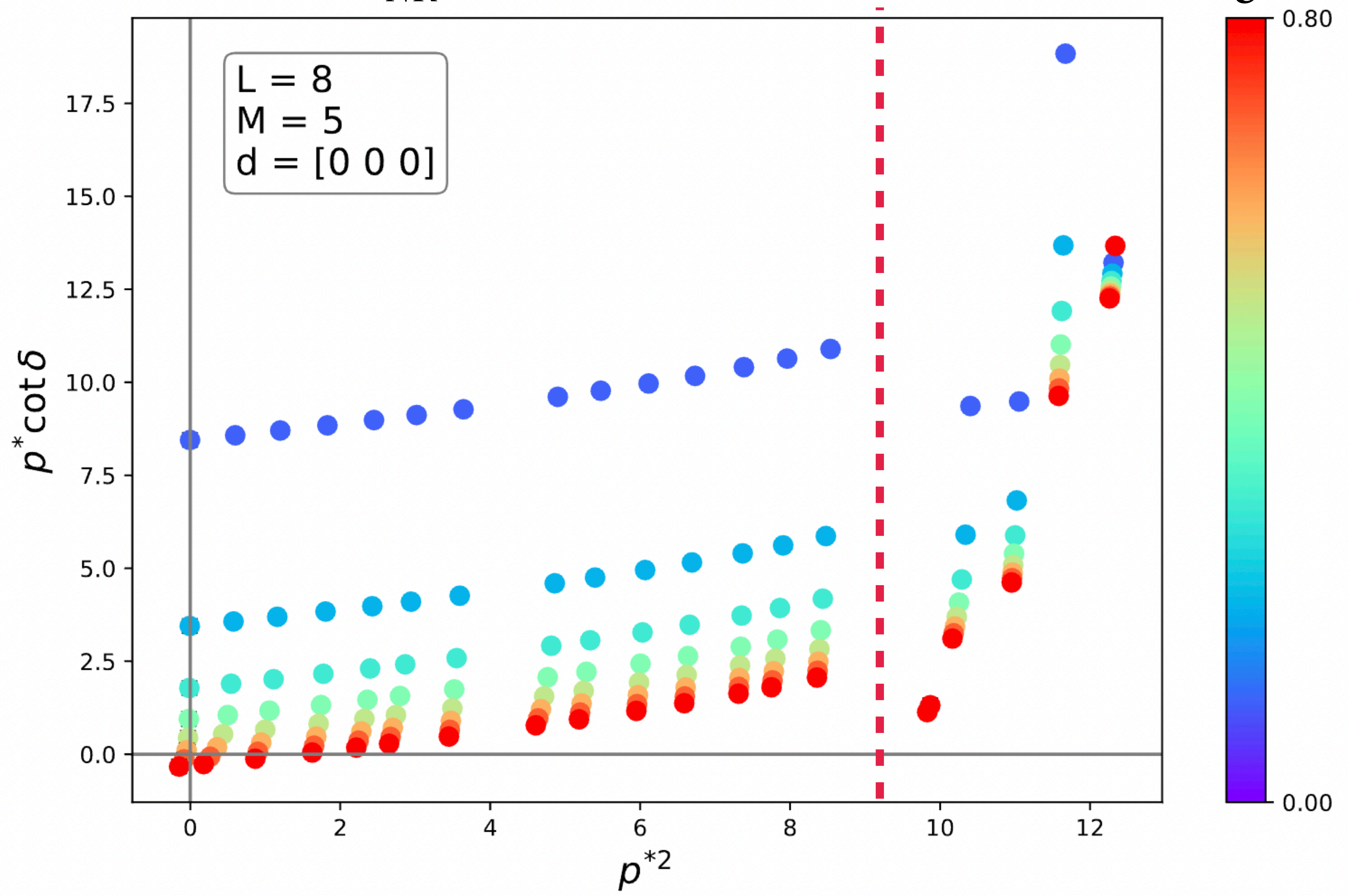
$$\mathcal{Z}_{00}[s; x^2] = \frac{1}{\sqrt{4\pi}} \sum_{\boldsymbol{n} \in \mathbb{Z}^3} \frac{1}{(|\boldsymbol{n}|^2 - x^2)^s}$$

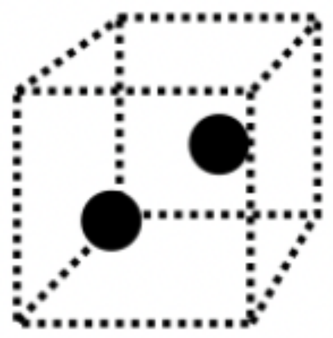


$p^* = ME^*_{NR}$

Discretization Effects

$O(p^2 a_s)$





Formalism is made for relativistic theory

No Physical Mass Scale in NR theory

$$M \sim 1/a_s$$

Assumption:

$m = M$

Extract E^* from E_{NR}

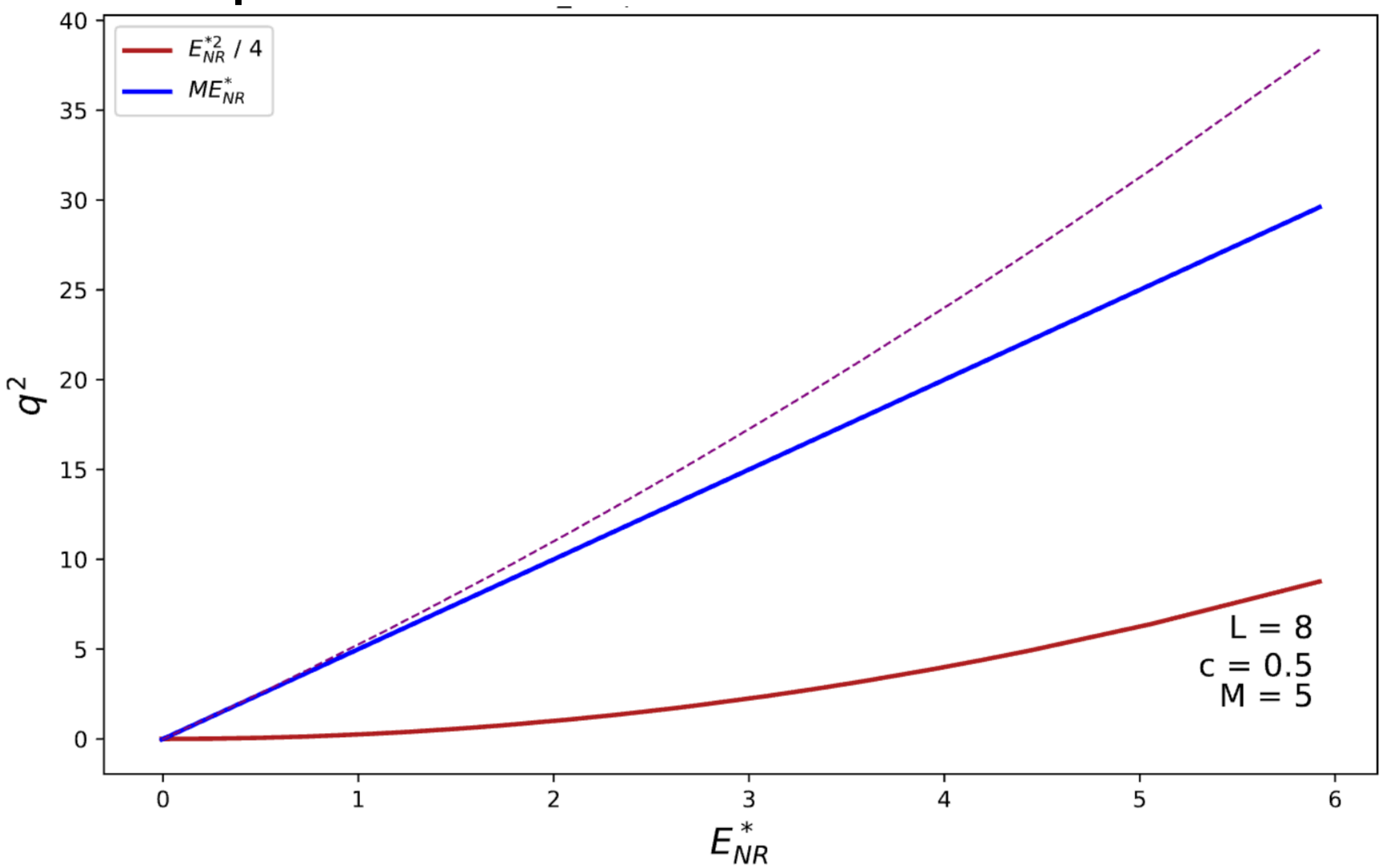
$$E^*_{NR} = E^* - 2m$$

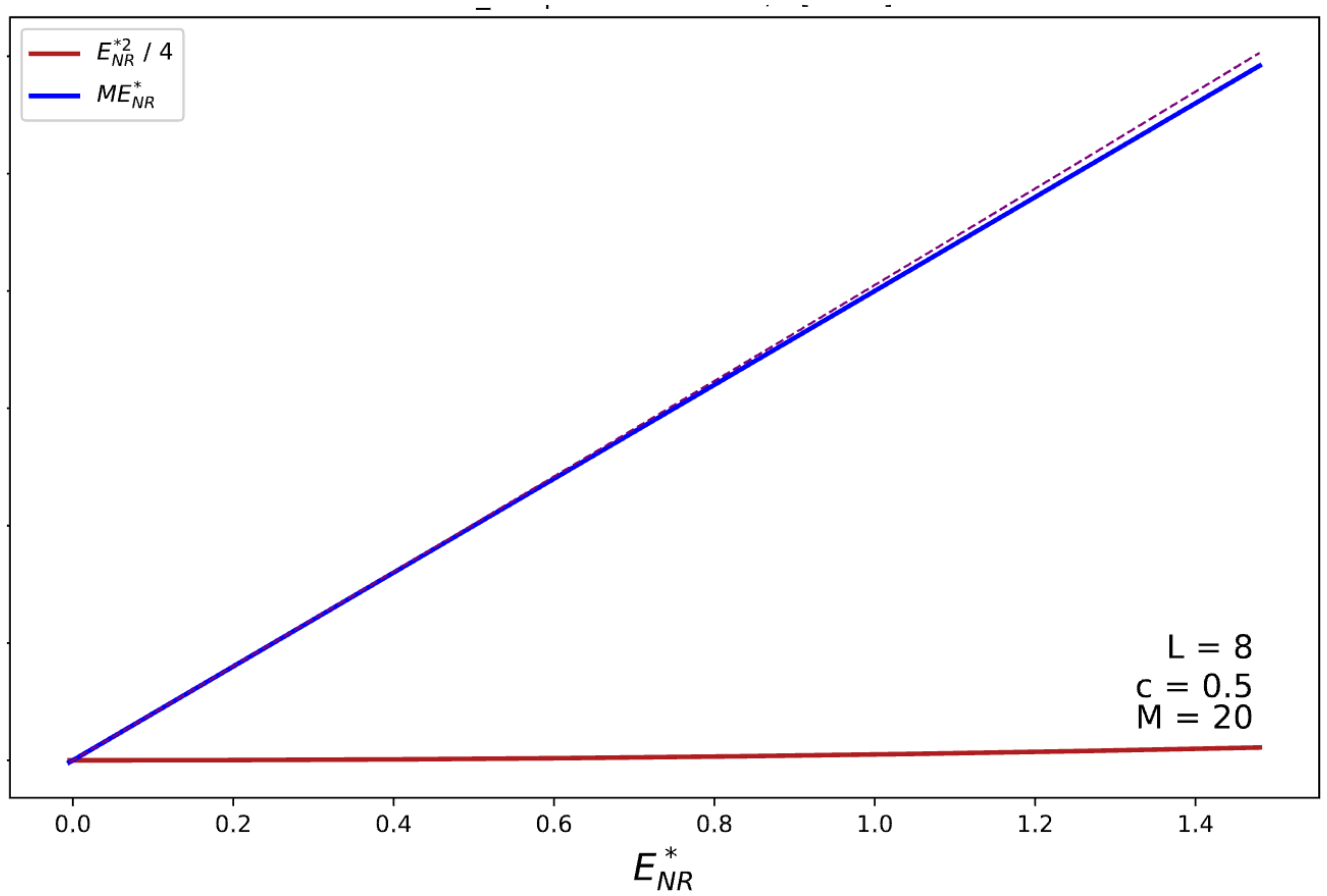
$$q^{*2} = \frac{E^{*2}}{4} - M^2$$

$$q^{*2} = \frac{E^{*2}_{NR}}{4} + ME^*_{NR}$$

$$M \to \infty$$

$$q^{*2} = ME_{NR}$$





Preliminary Results

$$\gamma = \frac{E}{E^*}$$

$$q^* \cot\delta(q^*) = \frac{2}{\gamma L\sqrt{\pi}} Z_{00}^{\mathbf{d}}(1; \frac{q^{*2}L^2}{4\pi^2})$$

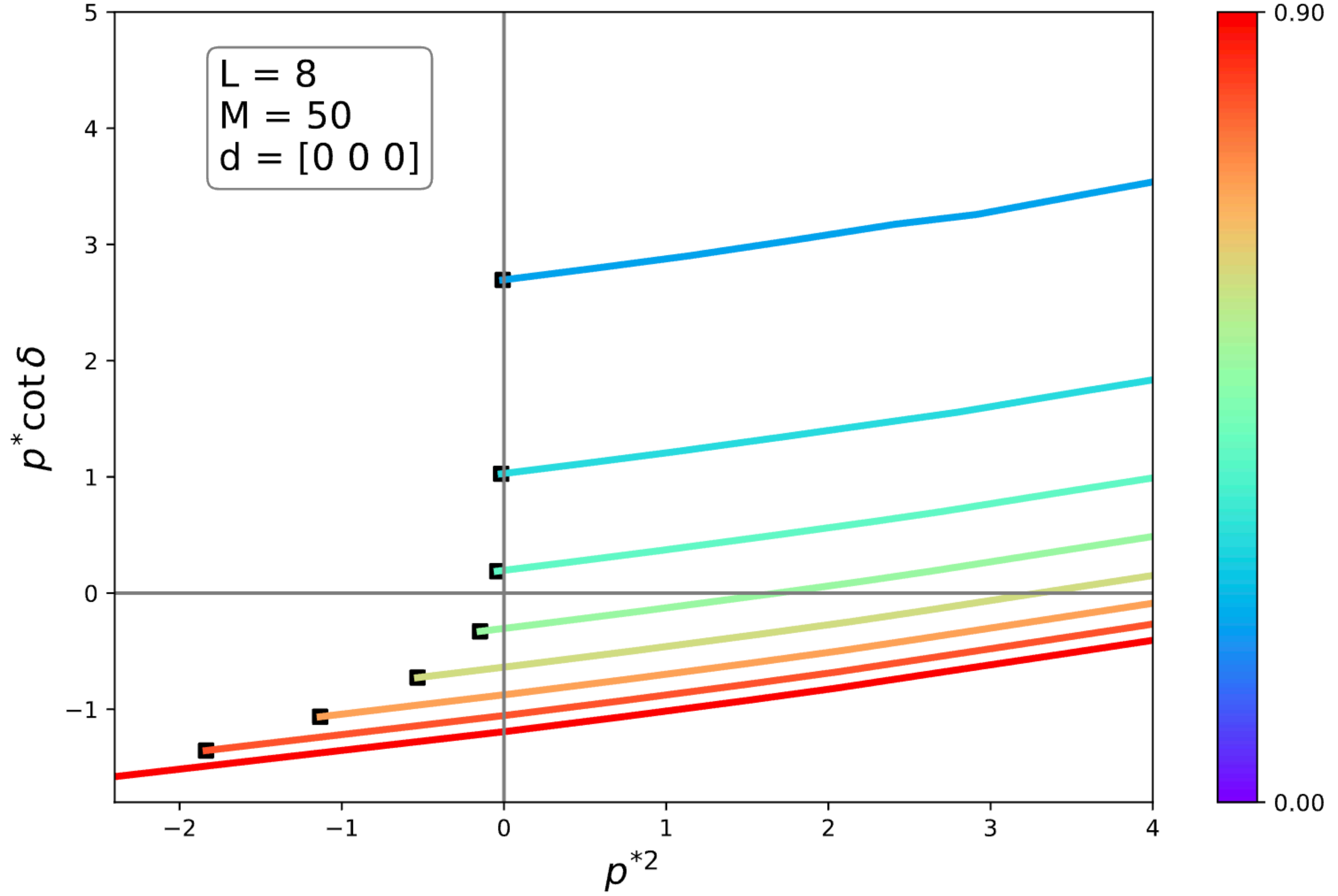
Assumption:

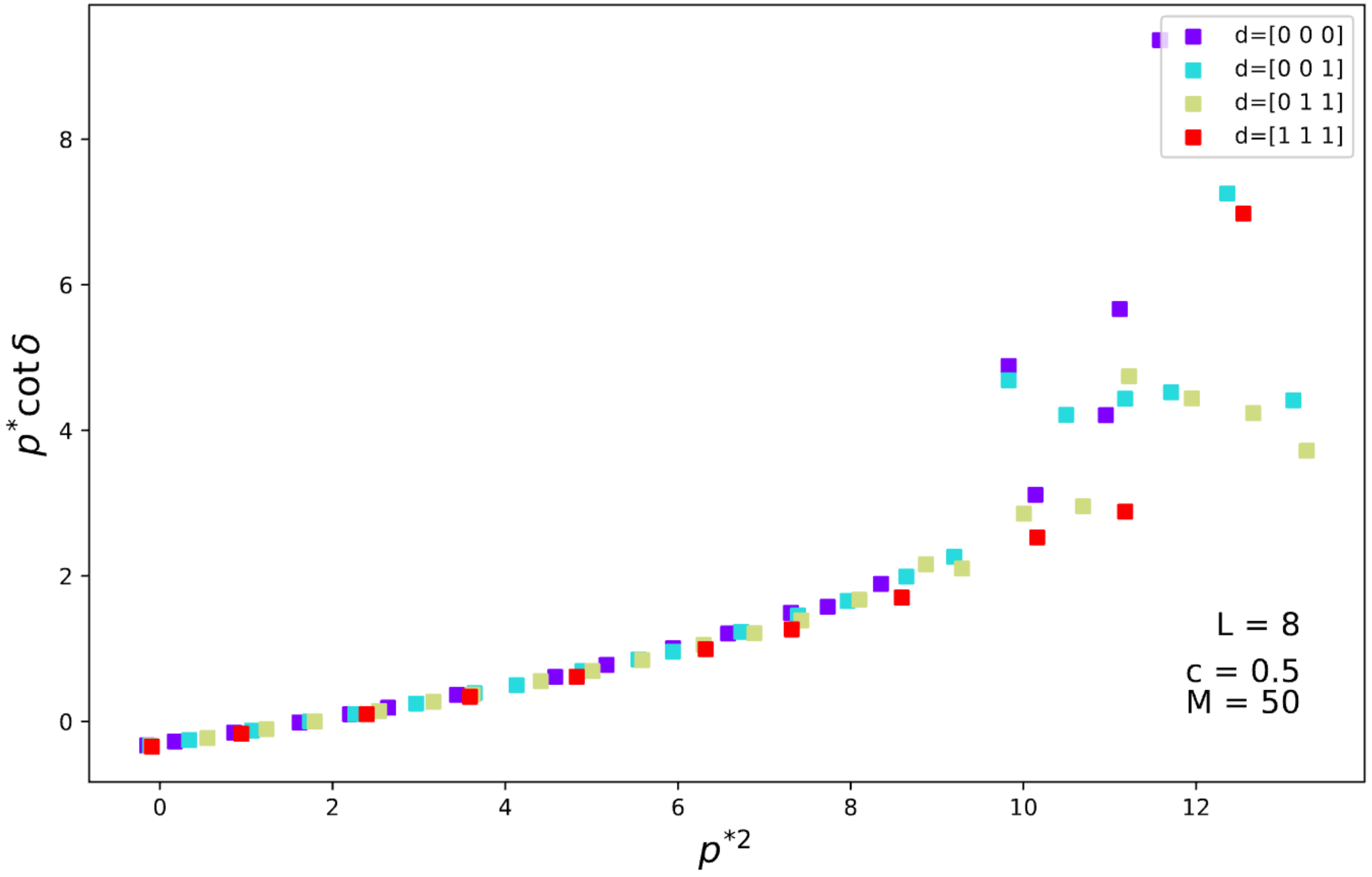
$m = M$

Relativistic

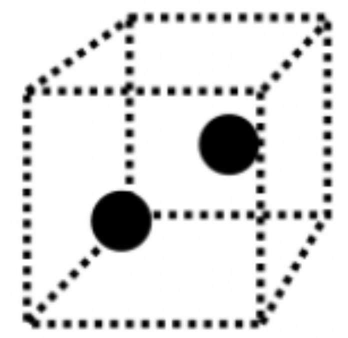
$M = 50$

- Extract E^* from E_{NR}
- $F(P, L)$ to get $p^* \cot\delta(p^*)$





Preliminary Results



- Extract E^* from E_{NR}

$$\gamma = \frac{E}{E^*}$$

$$q^* \cot \delta(q^*) = \frac{2}{\gamma L \sqrt{\pi}} Z_{00}^{\mathbf{d}}\left(1; \frac{q^{*2} L^2}{4\pi^2}\right)$$

Assumption:

$m = M$

Relativistic

$M = 50$

- $F(P, L)$ to get $p^* \cot \delta(p^*)$

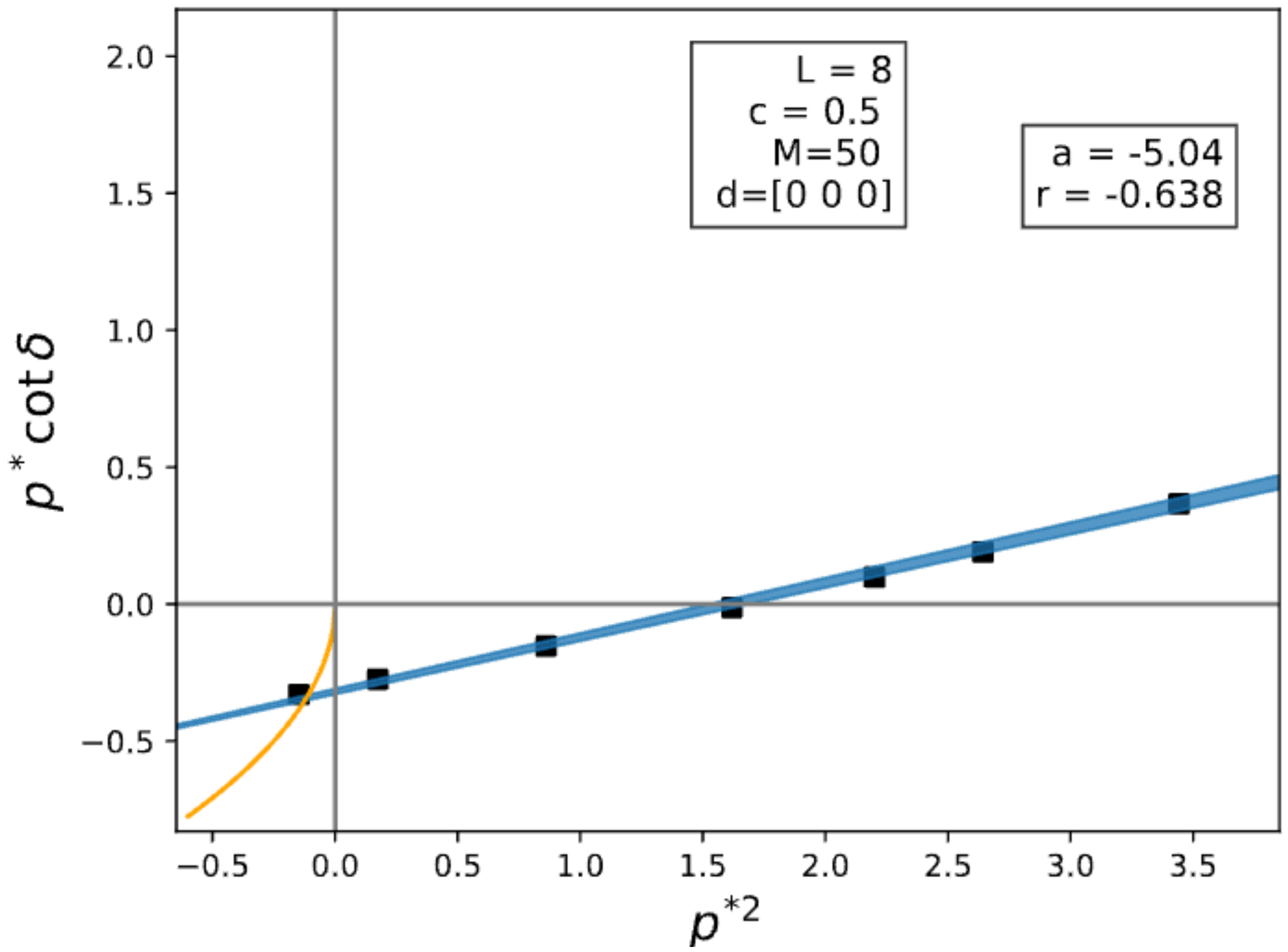
- Relativistic form of scattering amplitude

ERE fit

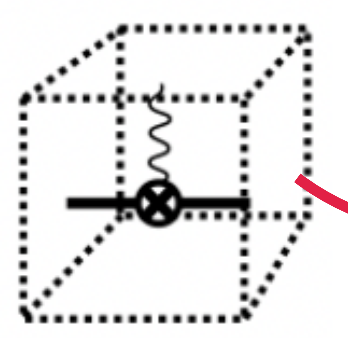
$$\mathcal{M}(s) = \frac{1}{\rho(s)\cot\delta(s) - i\rho(s)},$$

$$\rho(s) = \frac{q^\star}{8\pi E^\star}$$

$$q^\star \cot \delta^{\mathrm{NLO}}(s) = -\frac{1}{a} + \frac{1}{2} r q^{\star 2}$$



Preliminary Results

Single—Particle Form Factor

$$f(Q^2) = 1$$

$$Q^2 = -(P_f - P_i)^2$$

$$P = (E, \vec{P})$$

Ground-state NN system

Eigenvectors of Transfer Matrix

$$|(NN)_0(P)\rangle = \sum_i a_{k_i}(P)|p(k_i)\rangle|n(-k_i + P)\rangle$$

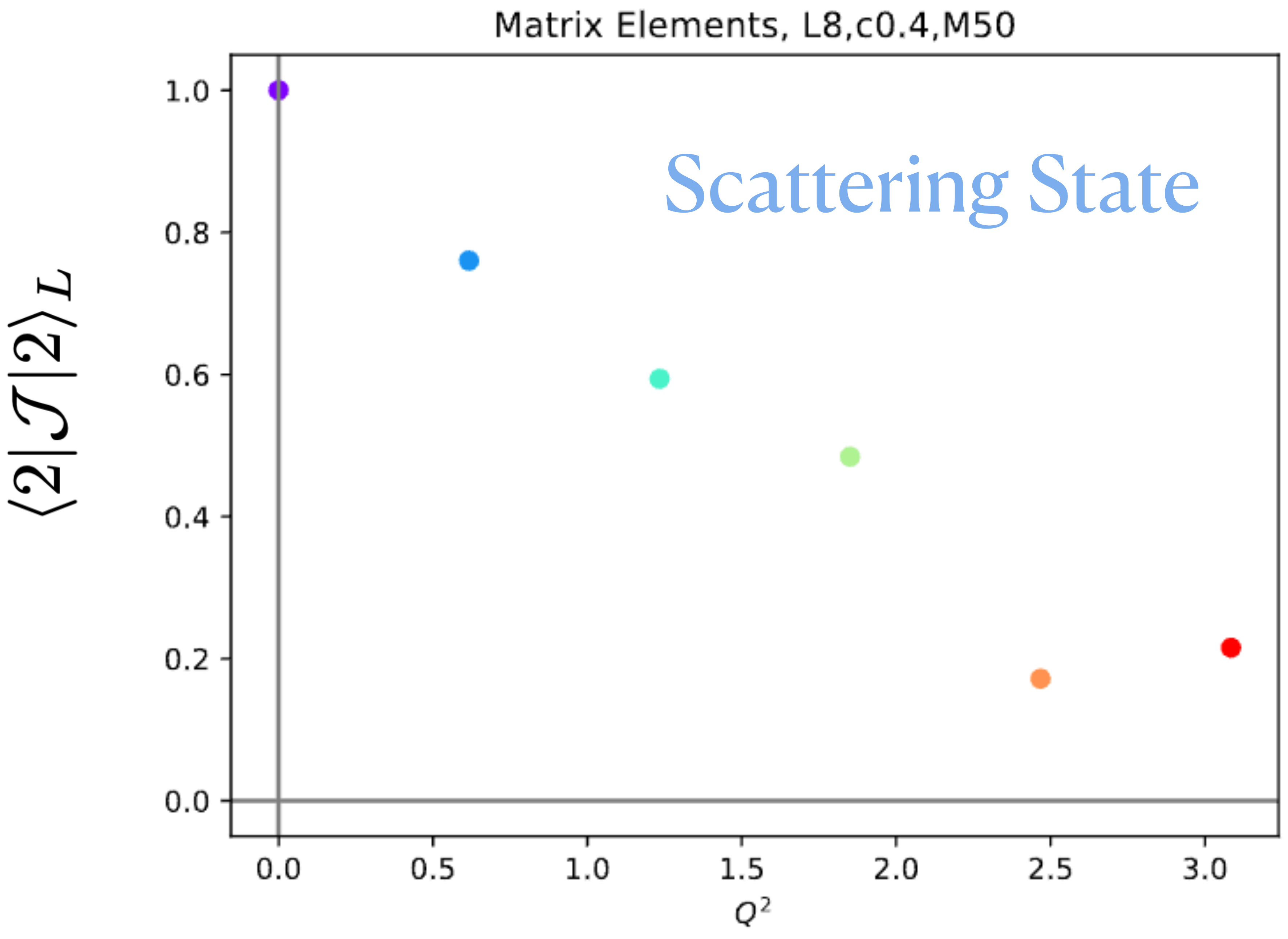
$$\langle(NN)_0(P)|J_1^\mu(Q)\,|(NN)_0(P)\rangle = \sum_m a_{k_m - Q}(P - Q)a_{k_m}(P)(2k_m^\mu - Q^\mu)$$

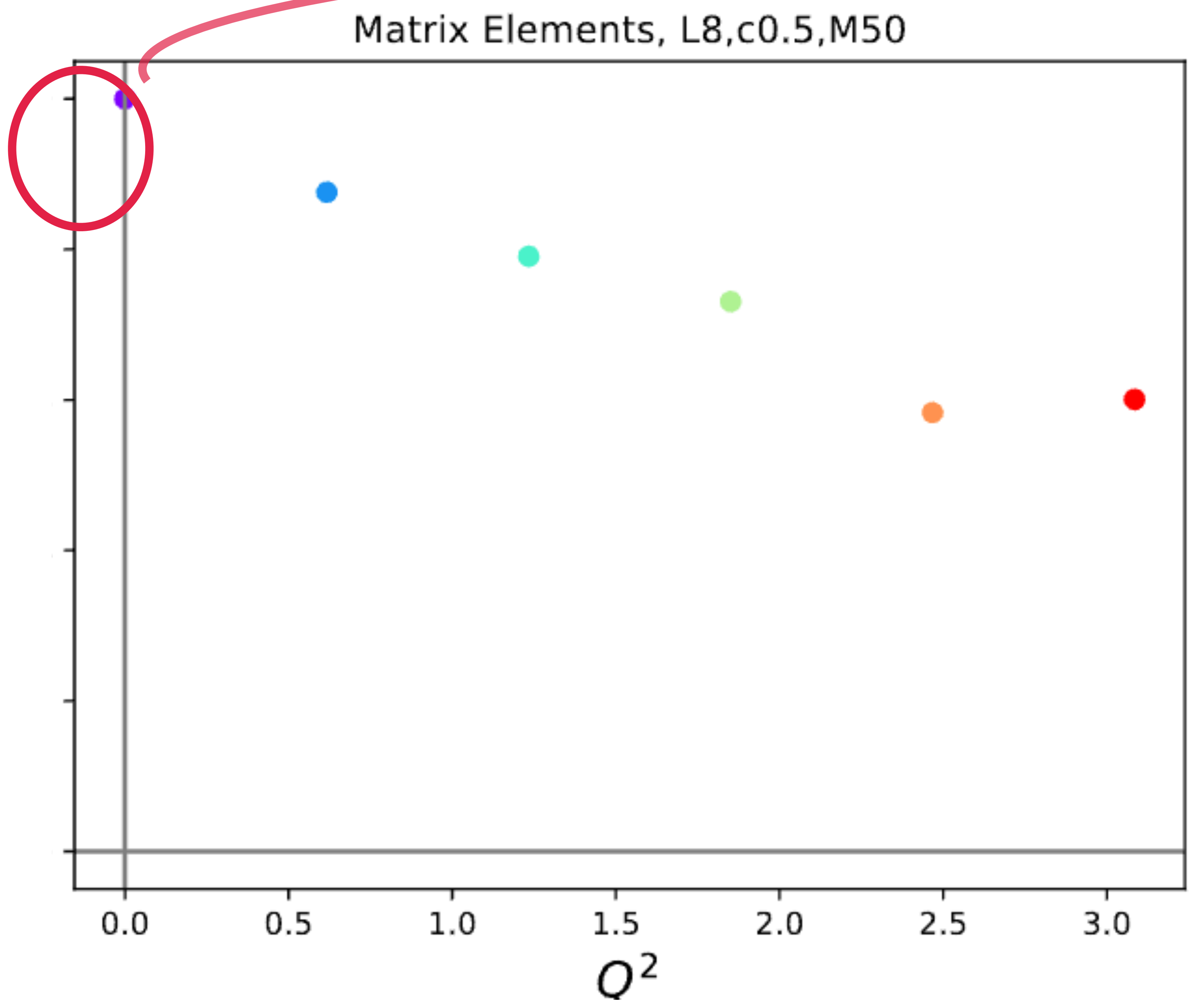
Timelike. $\mu = 0$

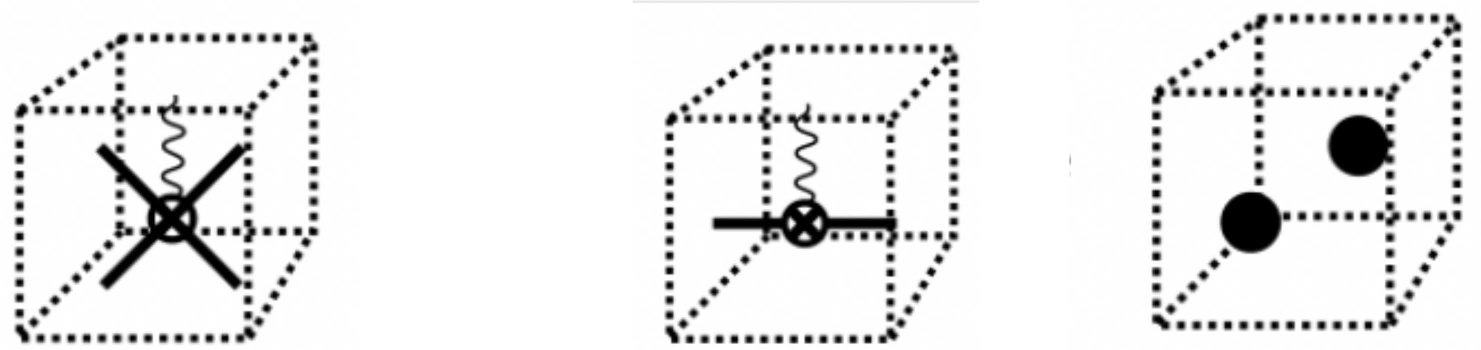
Current Conservation!

$$\langle P_n, L|\,\widehat{\mathrm{Q}}\,|P_n, L\rangle = \mathrm{Q}_0$$

$$\mathrm{Q}_0 = \mathrm{Q}_p + \mathrm{Q}_n$$







$$L^3 \langle P_{n,f}, L | \mathcal{J}^0 | P_{n,i}, L \rangle = \mathcal{W}^{0}_{L,\mathrm{df}}(P_{n,f}, P_{n,i}, L) \sqrt{\mathcal{R}(P_{n,f}, L)\mathcal{R}(P_{n,i}, L)}$$

Finite Volume Matrix Elements

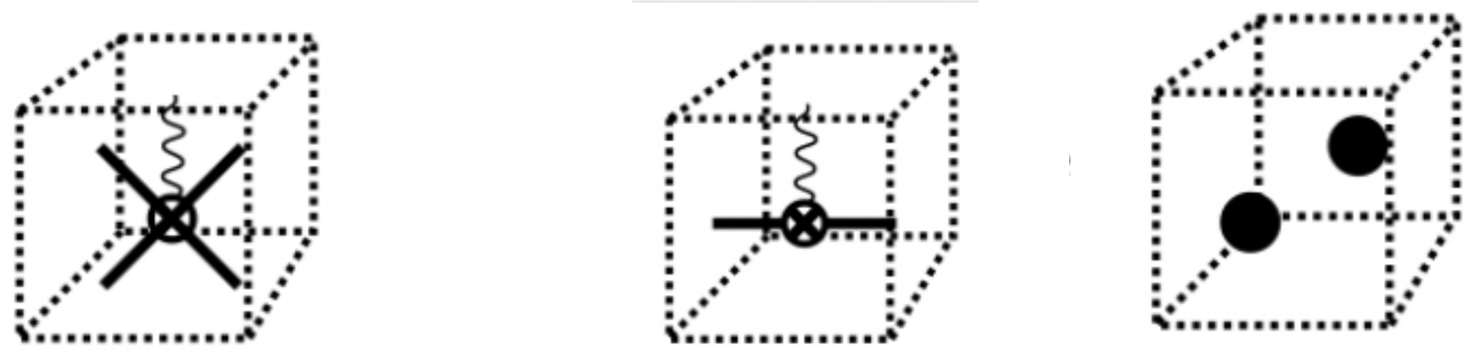
2+J->2 Matrix Element

LL factor

Correction of the FV states

Transition Amplitude

2+J->2 Matrix Element



LL factor

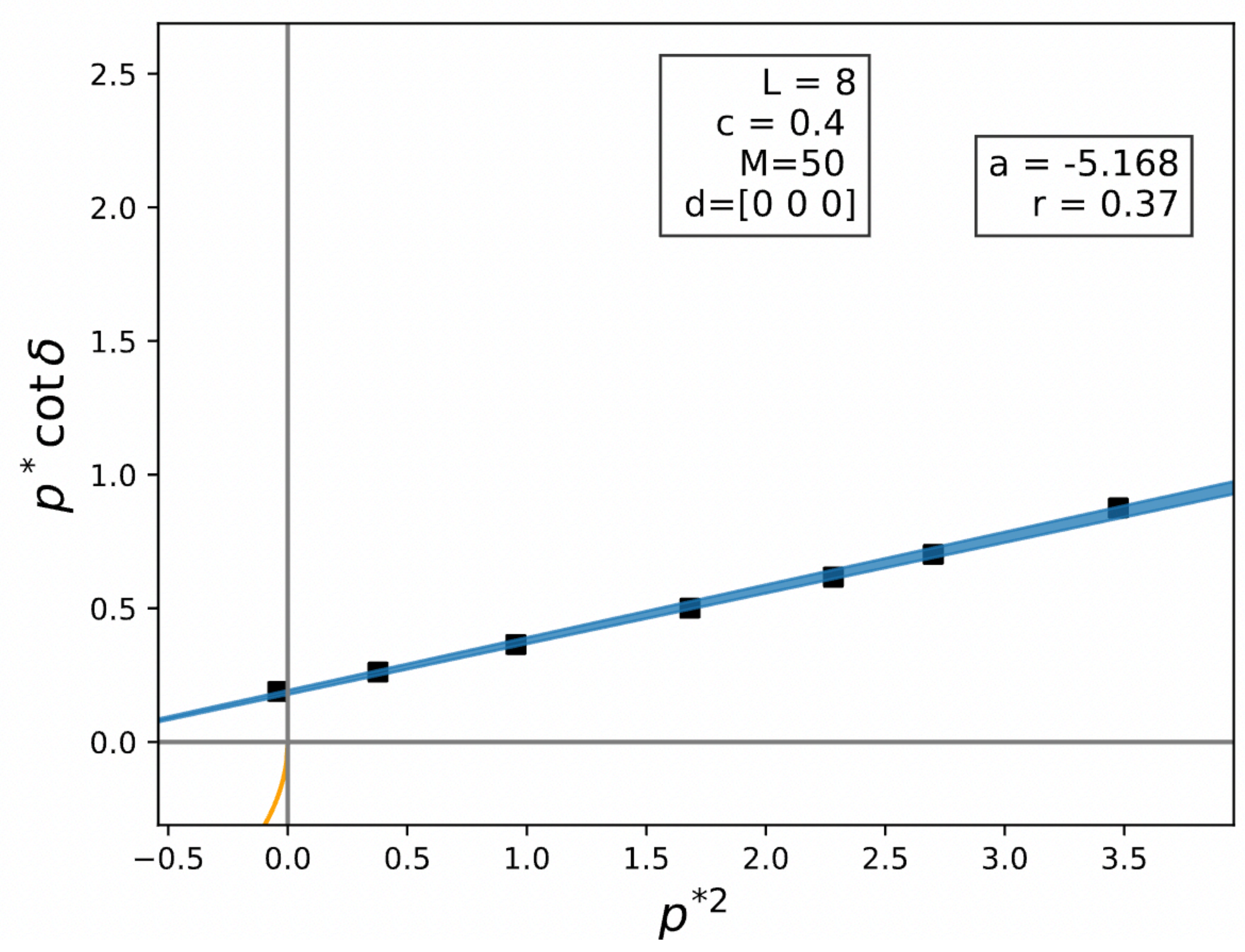
Correction of the FV states

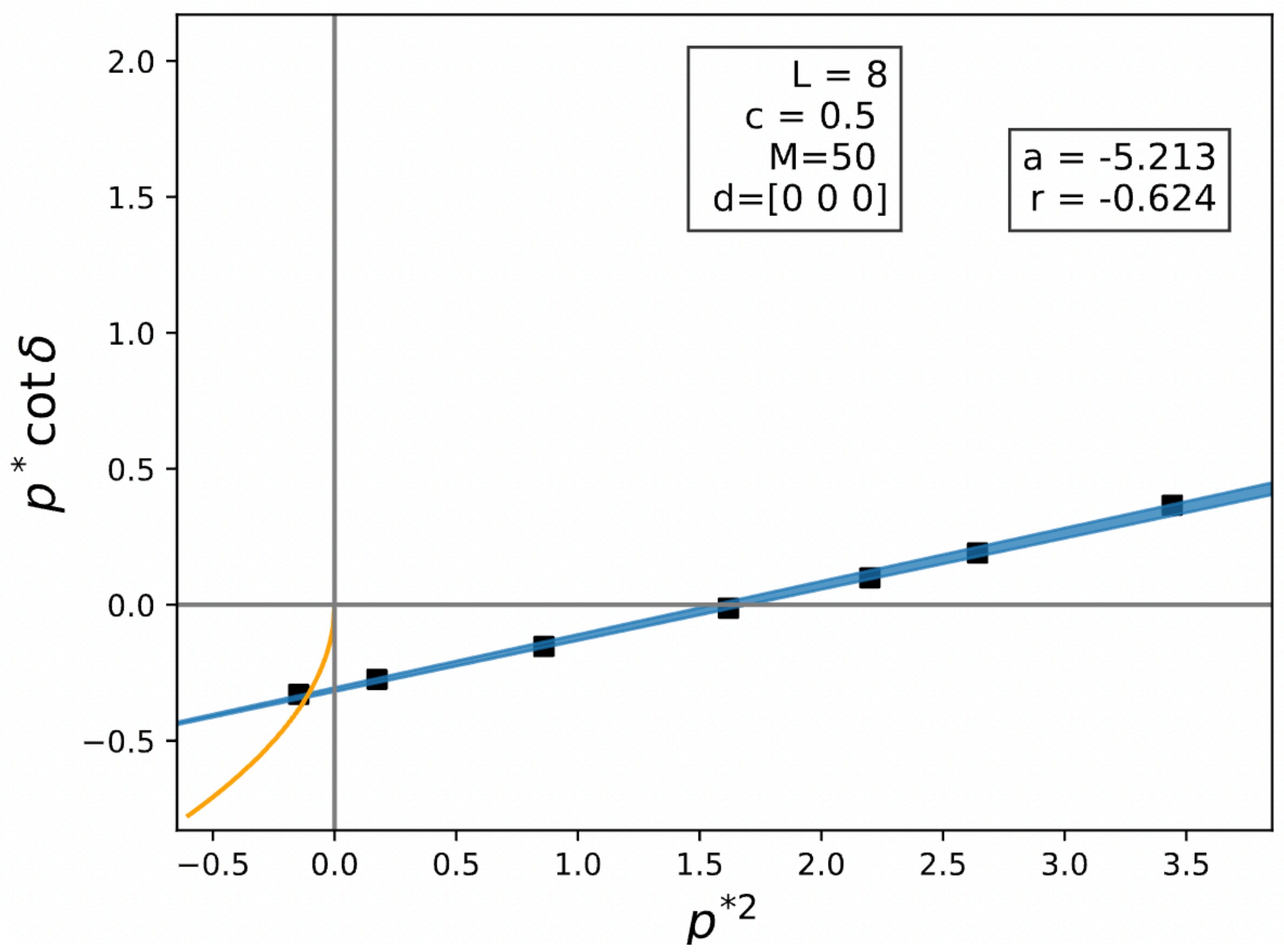
$$\mathcal{R}(P_n, L) = \left[\frac{\partial}{\partial E}\left(F^{-1}(P,L) + \mathcal{M}(s)\right)\right]^{-1}_{E=E_n} = \frac{1}{\mathcal{M}^2(s_n(L))}\left[-\frac{\partial}{\partial E}\mathcal{M}^{-1}(s) + 2E\,G(P,L) - 2G^{\mu=0}(P,L)\right]^{-1}_{P=P_n(L)}$$

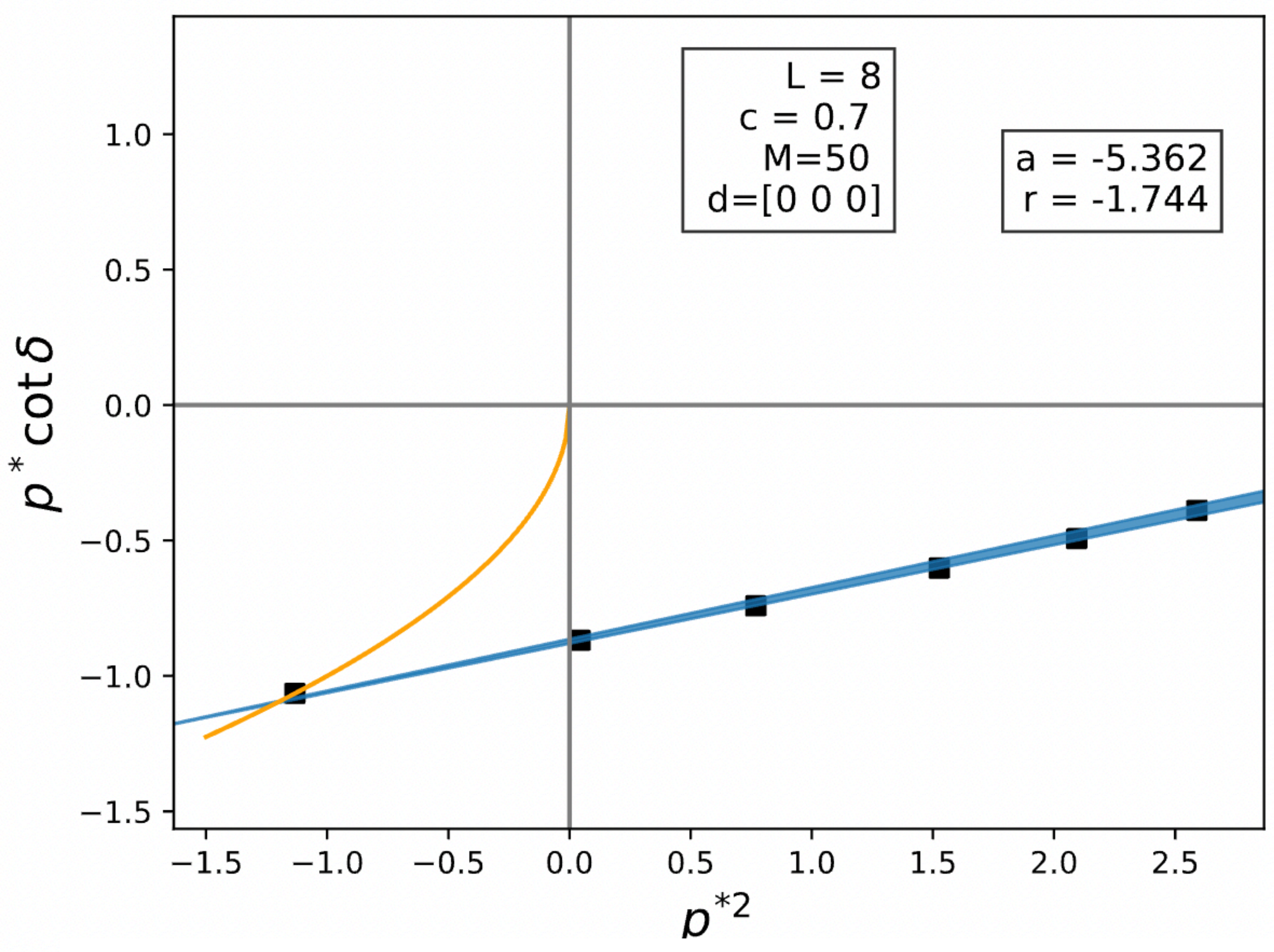
Scattering State

Near-Threshold Bound

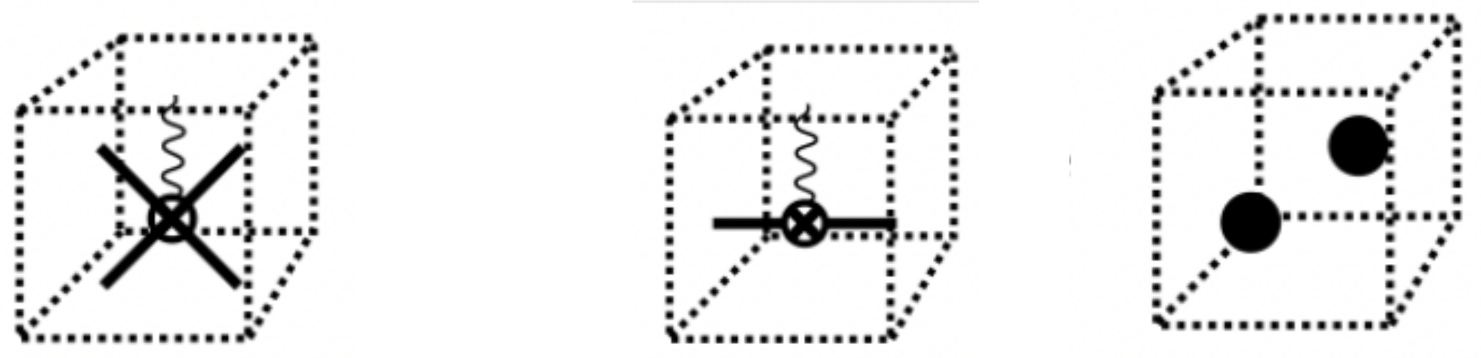
Deep Bound State







Preliminary Results

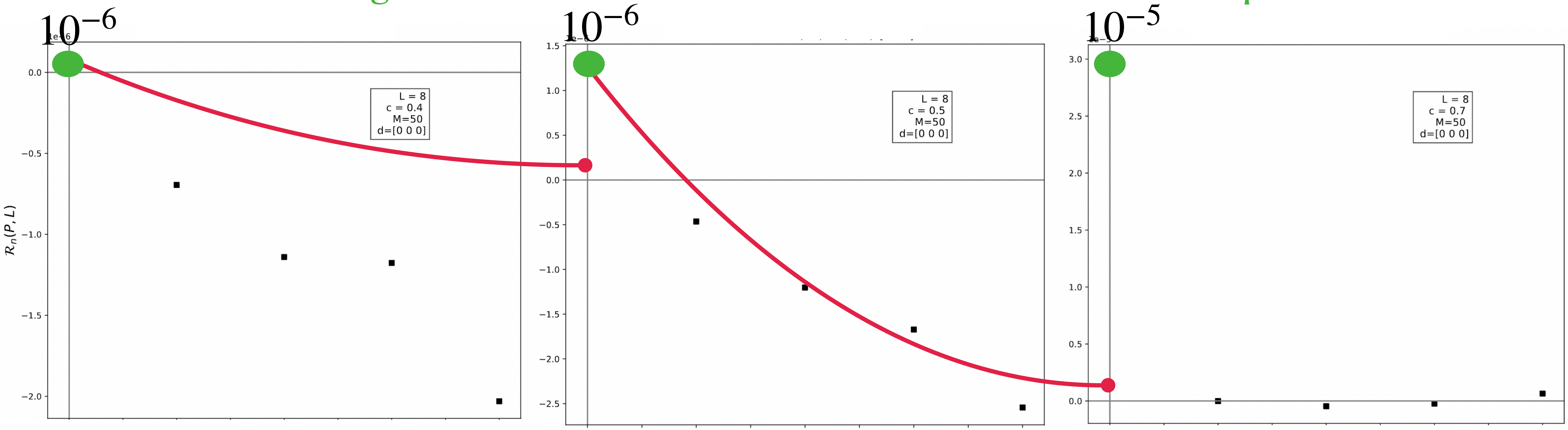


LL factor

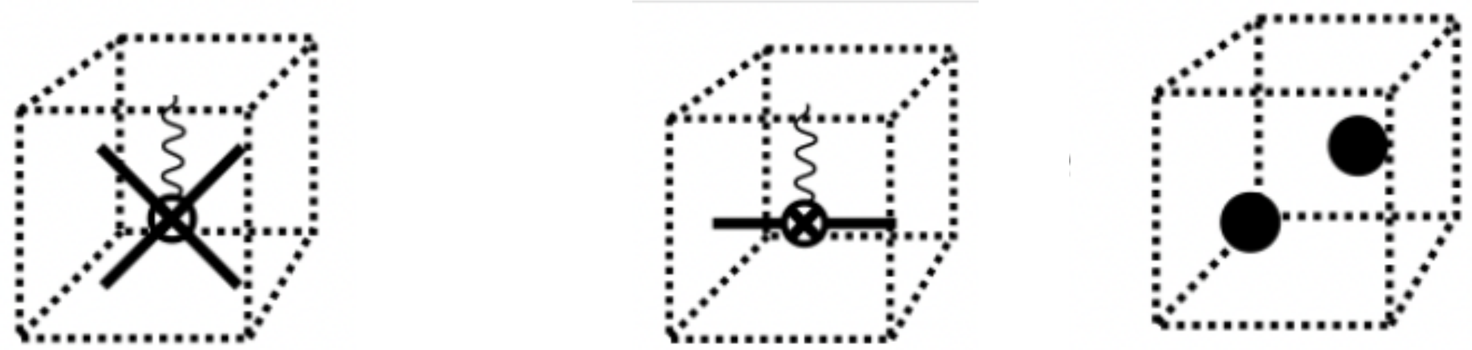
Correction of the FV states

$$\mathcal{R}(P_n, L) = \left[\frac{\partial}{\partial E}\left(F^{-1}(P,L) + \mathcal{M}(s)\right)\right]^{-1}_{E=E_n} = \frac{1}{\mathcal{M}^2(s_n(L))}\left[-\frac{\partial}{\partial E}\mathcal{M}^{-1}(s) + 2E\,G(P,L) - 2G^{\mu=0}(P,L)\right]^{-1}_{P=P_n(L)}$$

Scattering State | Near-Threshold Bound | Deep Bound State

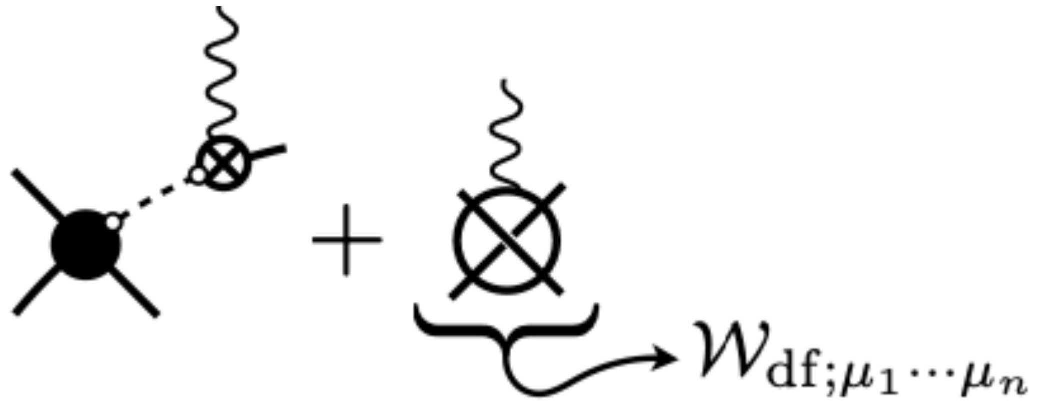


Preliminary Results



Divergence Free

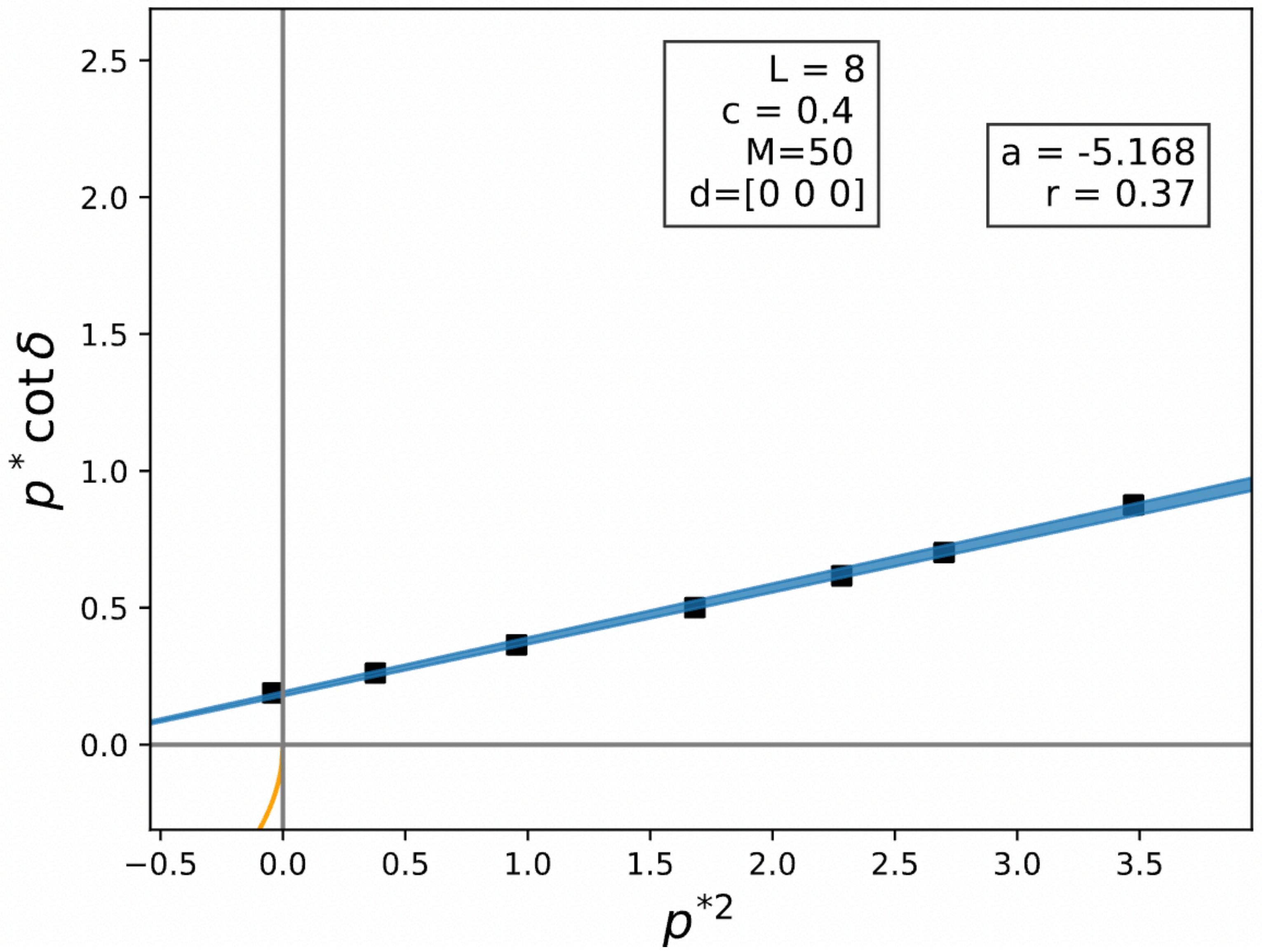
Free of 1+J->1 Poles



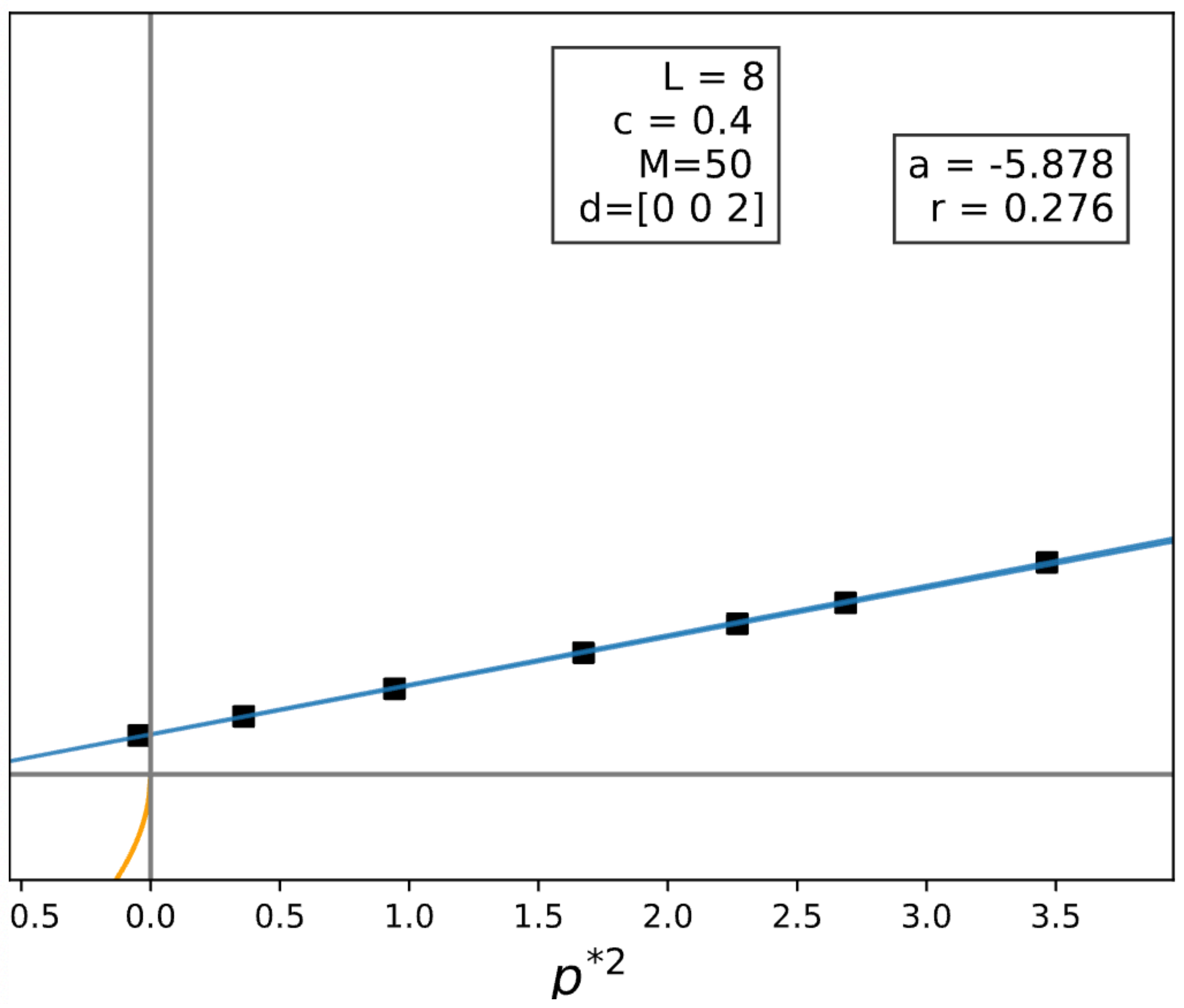
$$\mathcal{W}^{\mu}_{L,\mathrm{df}}(P_f,P_i,L) = \mathcal{W}^{\mu}_{\mathrm{df}}(P_f,P_i) + f(Q^2)\mathcal{M}(s_f)\Big[(P_f+P_i)^{\mu}G(P_f,P_i,L) - 2G^{\mu}(P_f,P_i,L)\Big]\mathcal{M}(s_i)$$

Scattering State **Fix L,c,M** **d = [0,0,0],[0,0,1],...** $Q^2 = -(P_f - P_i)^2$

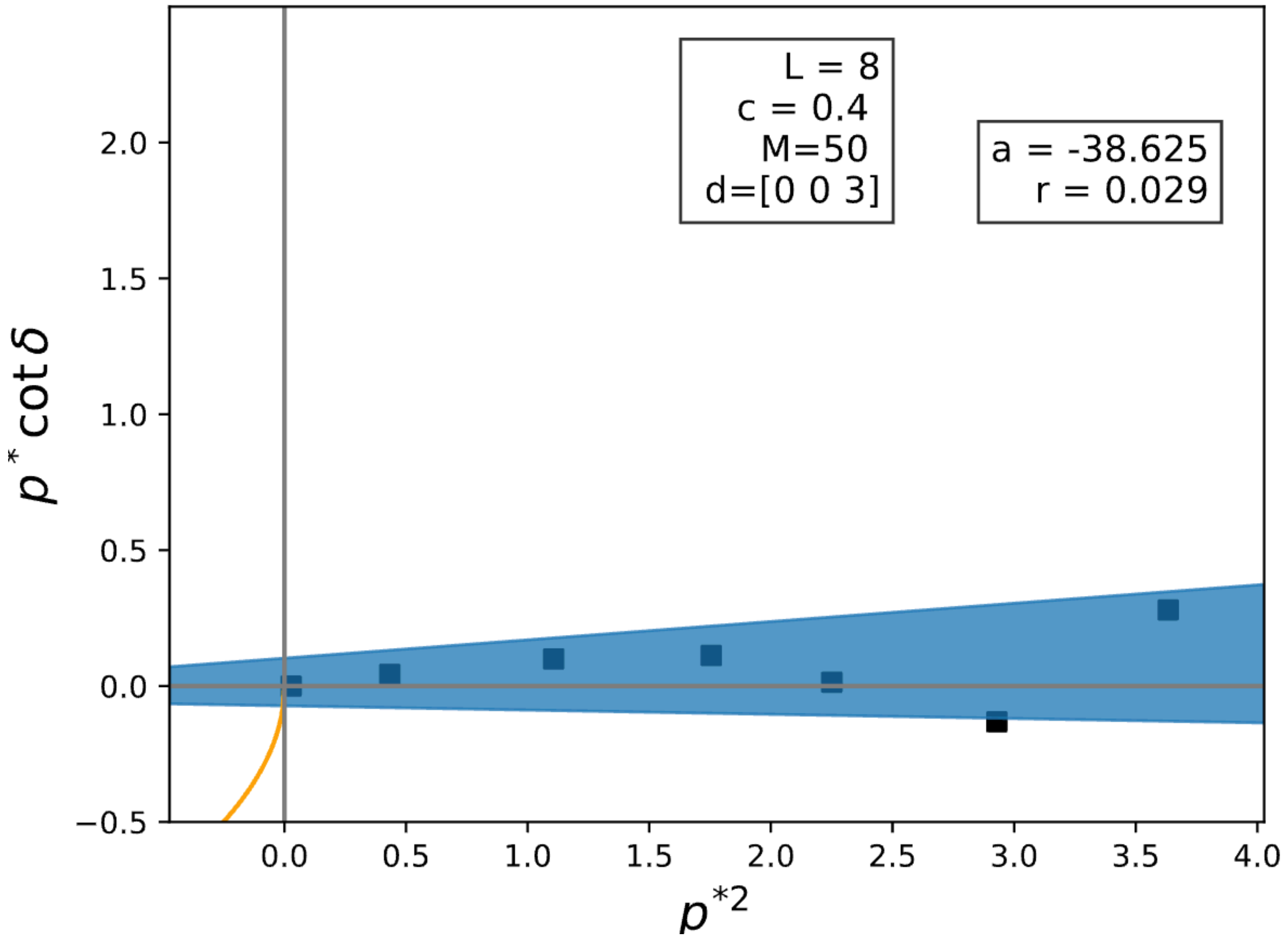
[0,0,0]



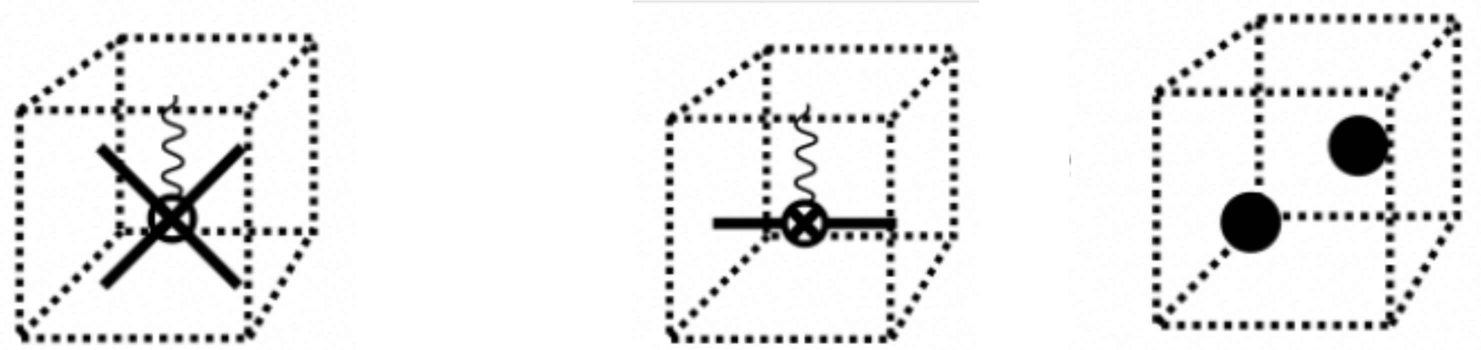
[0,0,2]



[0,0,3]

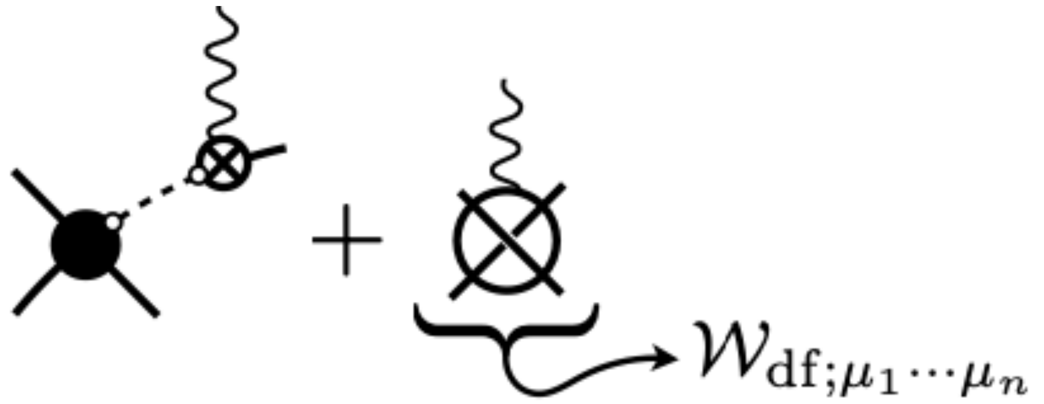


Preliminary Results



Divergence Free

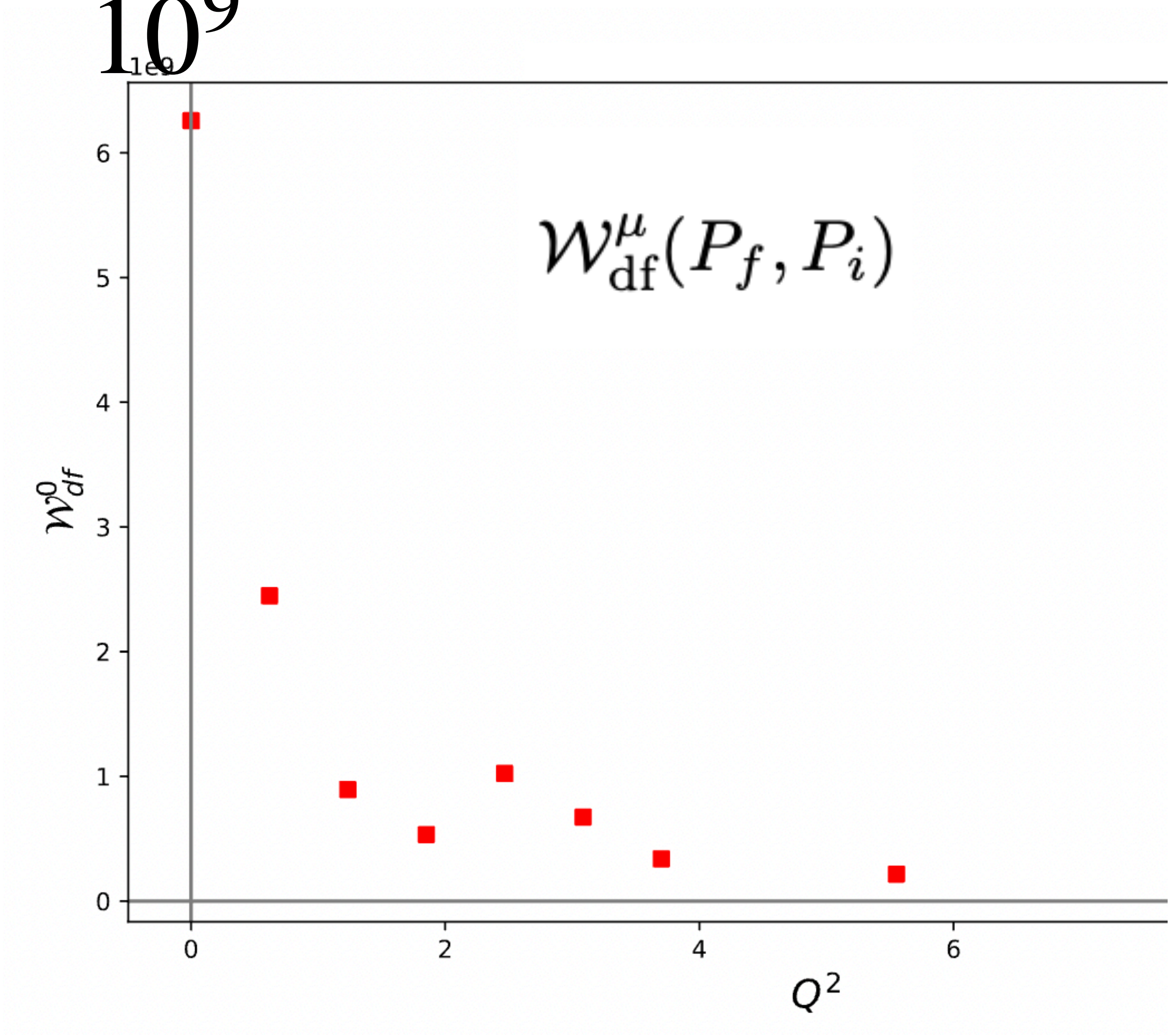
Free of 1+J->1 Poles



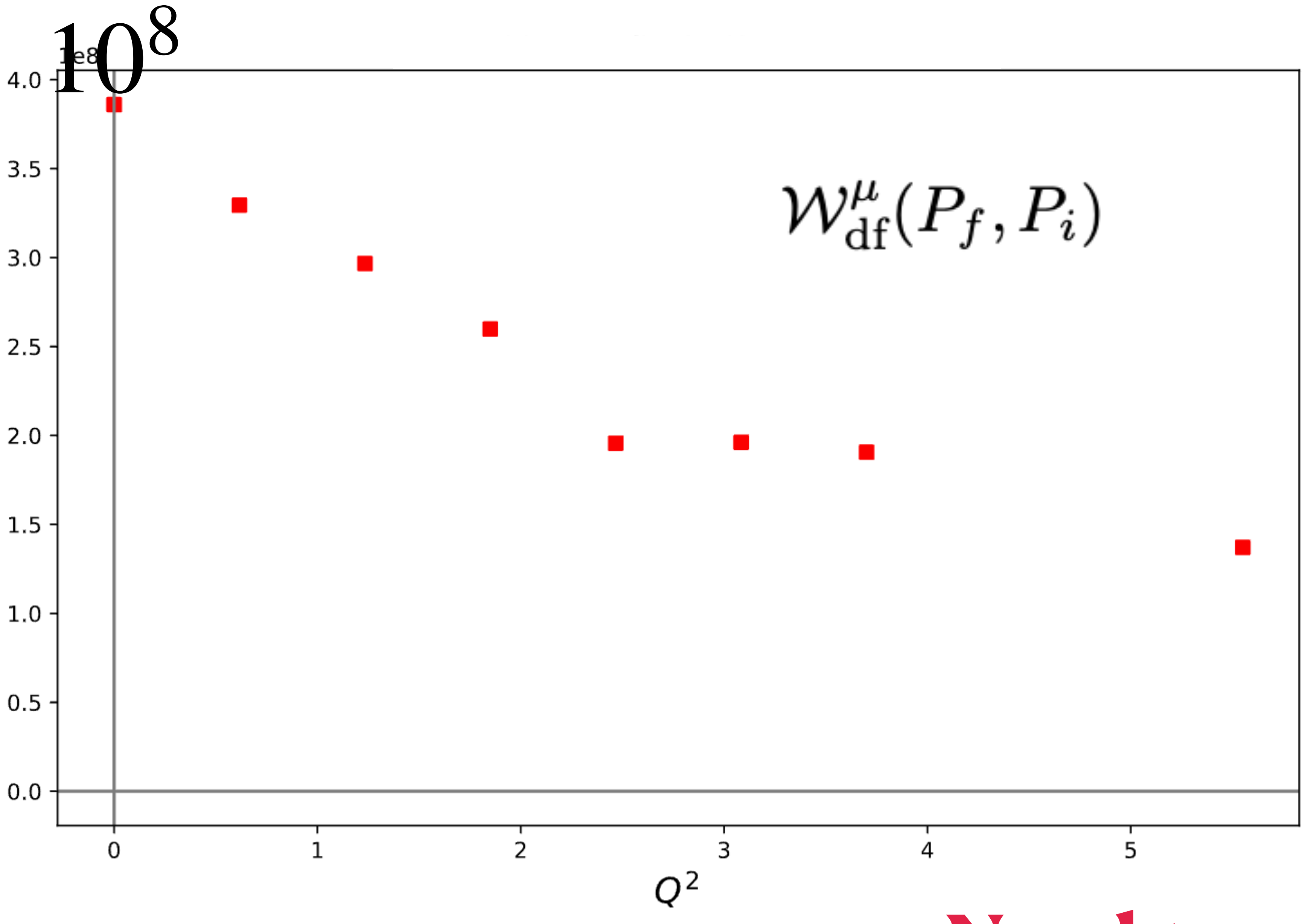
$$L^3 \langle P_{n,f}, L | \mathcal{J}^\mu | P_{n,i}, L \rangle = \mathcal{W}^\mu_{L,\mathrm{df}}(P_{n,f}, P_{n,i}, L)\sqrt{\mathcal{R}(P_{n,f}, L)\mathcal{R}(P_{n,i}, L)}$$

$$\mathcal{W}^\mu_{L,\mathrm{df}}(P_f, P_i, L) = \mathcal{W}^\mu_{\mathrm{df}}(P_f, P_i) + f(Q^2)\mathcal{M}(s_f)\Big[(P_f + P_i)^\mu G(P_f, P_i, L) - 2G^\mu(P_f, P_i, L)\Big]\mathcal{M}(s_i)$$

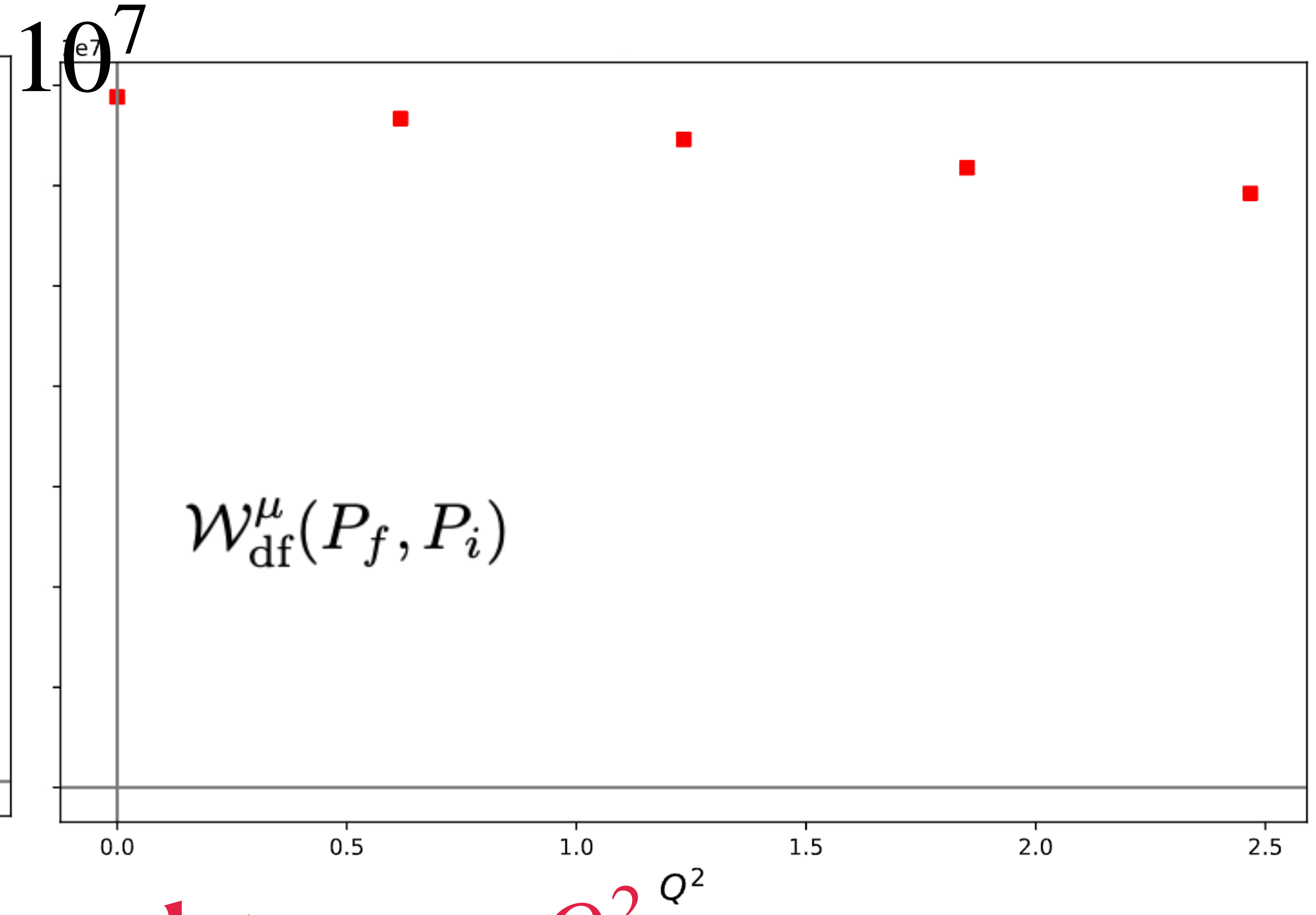
Scattering State



Near-Threshold Bound

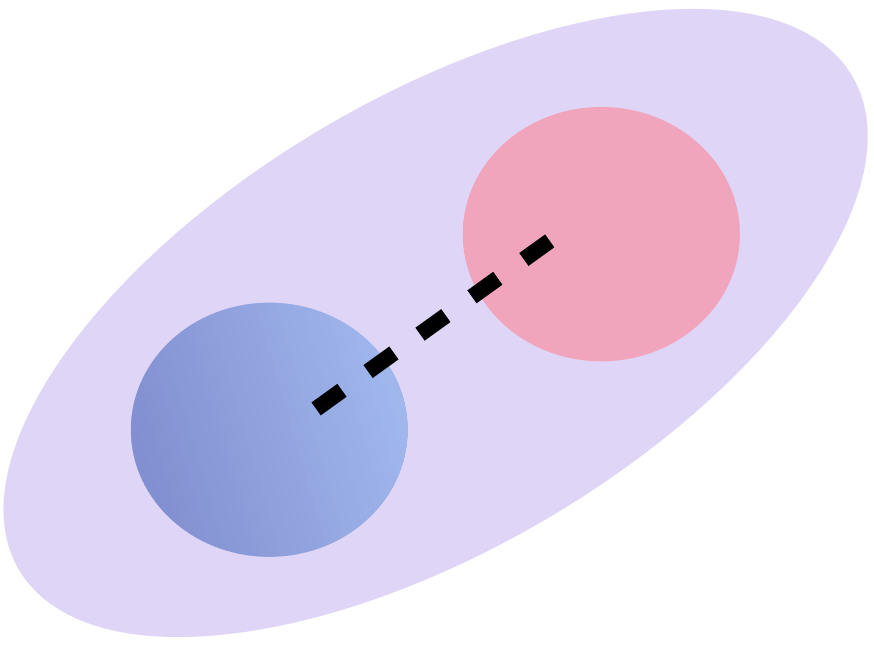


$\mathcal{W}^\mu_{\mathrm{df}}(P_f, P_i)$ Deep Bound State



Need to populate more Q^2

Conclusion



S-wave **bound-state**

Deuteron Break Up?

$$\mathbf{1} + \mathcal{J} \rightarrow \mathbf{2}$$

Reverse Reaction

$$\mathbf{2} + \mathcal{J} \rightarrow \mathbf{1}$$

- $\langle 2|\mathcal{J}|2\rangle_L$

  Formalism has been developed and will be critical in providing theoretical predictions of electroweak matrix elements from QCD.

- Preliminary Lattice EFT calculations of transition amplitude in three cases:

  Scattering State, Shallow Bound State, Deep Bound State

- **Next Steps:**

  Non-Relativistic Implementation of the Formalism (working out the kinks)

  Error Estimation through Gaussian Bootstrap

  Confirm Transition Amplitude behavior for Bound States

Acknowledgements

Berkeley

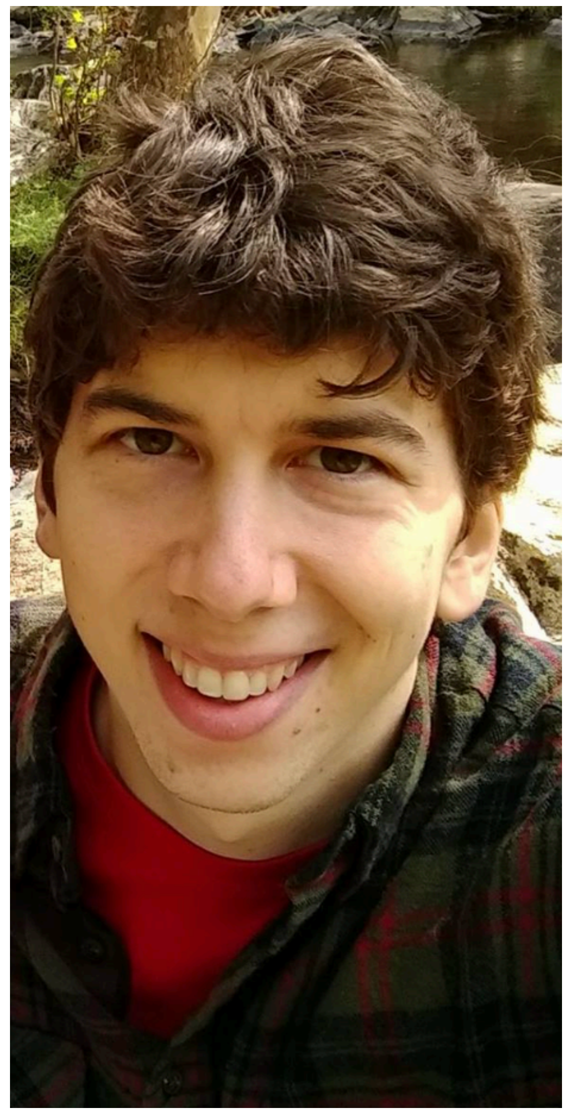
Raul Briceño

WILLIAM & MARY
CHARTERED 1693

Andrew Jackura

THE UNIVERSITY of NORTH CAROLINA at CHAPEL HILL

Amy Nicholson



Charles Kacir

Joseph Moscoso