The hadronic light-by-light contribution to the muon $g-2$ using staggered fermions at the physical point

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for the BMW collaboration

Centre de Physique Théorique Marseille

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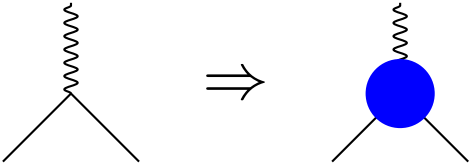
Content

Content

The anomalous magnetic muon moment g_μ

- For Dirac fermions $g_\mu = 2$ at the tree-level
- Deviation by quantum corrections of the fermion photon vertex:



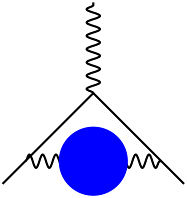
- Corrections quantified by $a_\mu = (g_\mu - 2)/2$
- The magnetic moment of the muon can be determined precisely in the experiment as well as in theory
- Sensitive to new physics
- **Recent experimental value:**
  $a_\mu = 116\ 592\ 055(24) \times 10^{-11}$ (Fermilab, 2023)
  [Phys. Rev. Lett. 131, 161802 (2023)]
- More precise measurements are expected in the near future $\Rightarrow$ error on the theory side needs to be reduced as well

Theoretical determination of a_μ

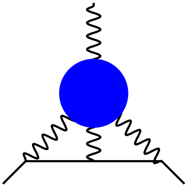
Standard Model contributions and current state results (see "white paper" from $g-2$ theory initiative [arXiv:2006.04822]):

contrib	$a_\mu \times 10^{11}$
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
LO-HVP (pheno)	6845(40)
LO-HVP (BMW '24)	7141(33)
HLbL (pheno & latt)	92(18)
total SM (WP)	116 591 810(43)

SM uncertainties dominated by two types of hadronic contributions:



LO-HVP
$\mathcal{O}(\alpha^2)$



HLbL
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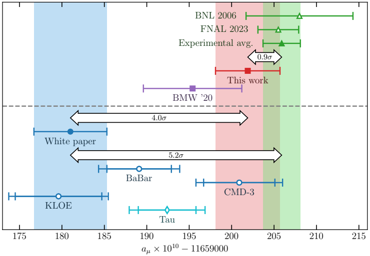
Tension between experiment and theory?

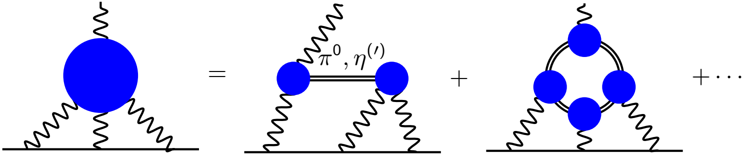


[BMW '24]

Determination of hadronic contributions

HLbL: Two approaches:

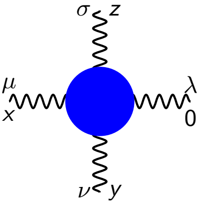
- **Dispersive Method***[Colangelo et al '17, Hoferichter et al '18]*:



  - Data driven approach, input by lattice calculations *[Mainz '19, ETM '23, BMW '23]*

- **Evaluate 4pt function on the lattice** **(this work)**

$$\widetilde{\Pi}_{\mu\nu\lambda\sigma}(x,y,z) =$$

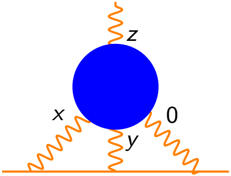


$$= \langle j_\mu(x) j_\nu(y) j_\lambda(0) j_\sigma(z)\rangle$$

  - Need precision of 10% to match the future experimental precision
  - BUT: 4pt functions are in general challenging on the lattice

Content

HLbL contributions to a_μ with staggered fermions



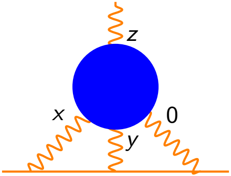
HLbL contributions to a_μ with staggered fermions

Master formula for hadronic light-by-light contribution to a_μ *[Mainz '21, 22']*:

$$a_\mu^{\mathrm{HLbL}} = \frac{m_\mu e^6}{3} \int_{x,y,z} \mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\ (-z_\rho)\ \widetilde{\Pi}_{\mu\nu\sigma\lambda}(x,y,z)$$

- $\widetilde{\Pi}_{\mu\nu\sigma\lambda}(x,y,z) := \langle j_\mu(x) j_\nu(y) j_\sigma(z) j_\lambda(0)\rangle$: with vector currents $j_\mu(x)$
- $\mathcal{L}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$: QED-kernel (not unique) *[RBC/UKQCD '17, Mainz '20]*
- 12-dimensional integral, 8 can be evaluated directly in the simulation
- Remaining object is rotational invariant $\Rightarrow$ Integrate over $|y|$

$$a_\mu^{\mathrm{HLbL}} = \frac{m_\mu e^6}{3} 2\pi^2 \int_{|y|} |y|^3 f(|y|)$$



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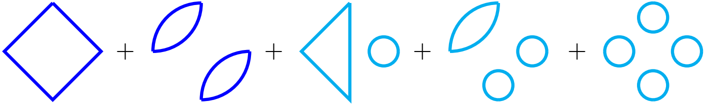
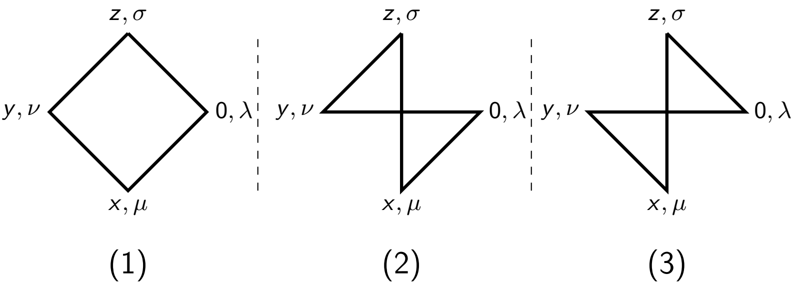
Decomposition of the HLbL tensor $\widetilde{\Pi}$ in terms of Wick contractions:

$\widetilde{\Pi} =$ 

Connected graphs



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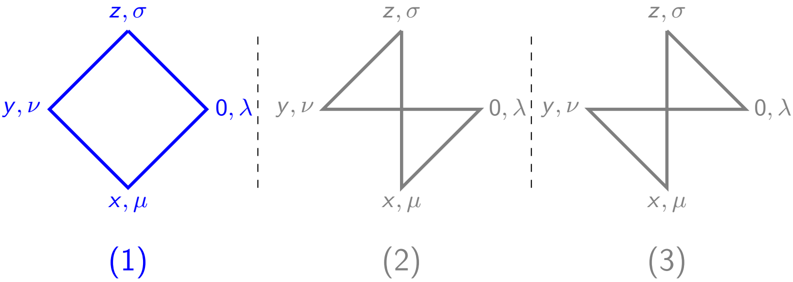

$$a_\mu^{\mathrm{conn}} = \frac{m_\mu e^6}{3} \sum_{x,y,z} \mathcal{L}^{(\mathrm{sym})}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\ (x_\rho - 3z_\rho)\,\mathrm{Re}\left\{\widetilde{\Pi}^{\mathrm{conn},(1)}_{\mu\nu\sigma\lambda}(x,y,z)\right\}$$

- Re-express contractions (2) and (3) in terms of (1) by exploiting translational invariance
- Fully symmetric kernel $(x \leftrightarrow y,\ y \leftrightarrow x-y)$ $\mathcal{L}^{(\mathrm{sym})}_{[\rho,\sigma];\mu\nu\lambda}(x,y)$ (different choice compared to Mainz)

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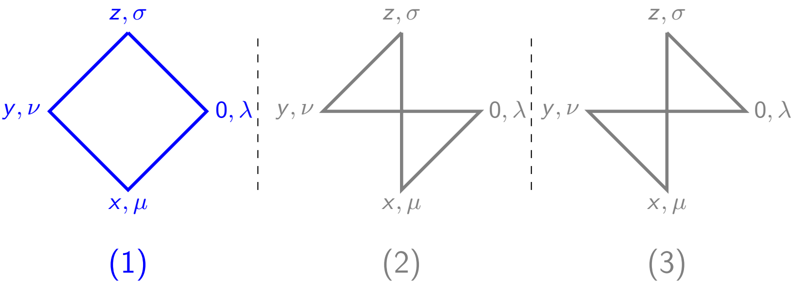

$$a_\mu^{\text{conn}} = \frac{m_\mu e^6}{3} 2\pi^2 \sum_{|y|} |y|^3 \underbrace{\sum_{x,z} \mathcal{L}^{(\text{sym})}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\,(x_\rho - 3z_\rho)\,\text{Re}\left\{\widetilde{\Pi}^{\text{conn},(1)}_{\mu\nu\sigma\lambda}(x,y,z)\right\}}_{=f(|y|)}$$

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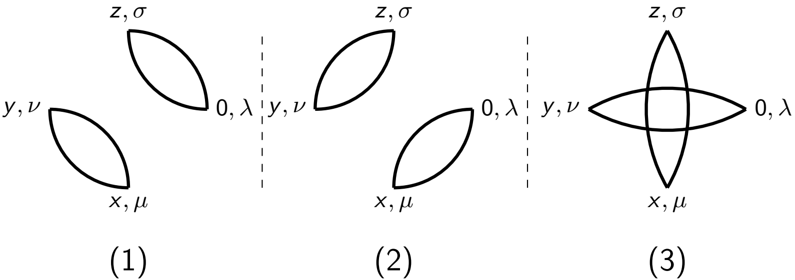
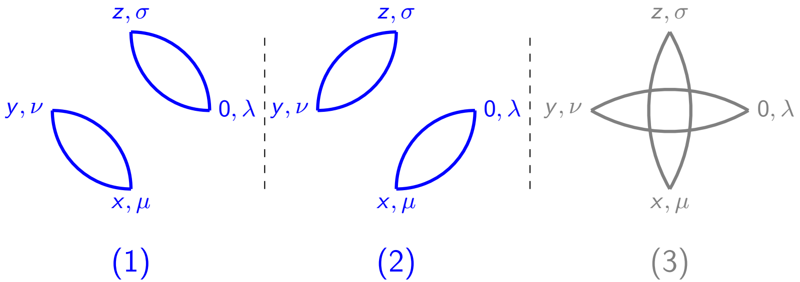
2+2 graphs



z, σ
y, ν
$0, \lambda$
x, μ
z, σ
y, ν
$0, \lambda$
x, μ
z, σ
y, ν
$0, \lambda$
x, μ
(1)
(2)
(3)

2+2 graphs



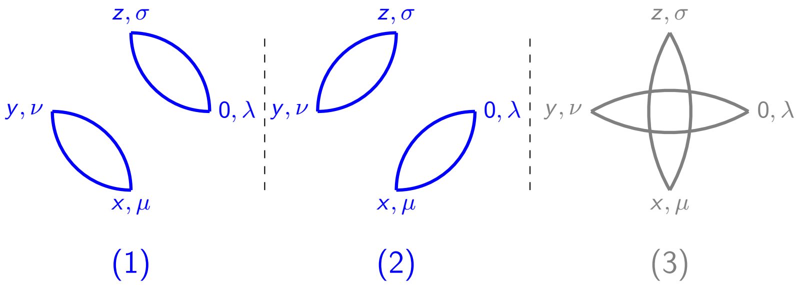
Two versions:

$$a_\mu^{(2+2),1} = \frac{m_\mu e^6}{3} 2\pi^2 \sum_{|y|} |y|^3 \sum_{x,z} \mathcal{L}^{(\mathrm{sym})}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\ (x_\rho + y_\rho - 3z_\rho)\ \widetilde{\Pi}^{(2+2),(1)}_{\mu\nu\sigma\lambda}(x,y,z)$$

$$a_\mu^{(2+2),2} = \frac{m_\mu e^6}{3} 2\pi^2 \sum_{|y|} |y|^3 \sum_{x,z} \mathcal{L}^{(\mathrm{sym})}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\ (y_\rho - 3z_\rho)\ \widetilde{\Pi}^{(2+2),(2)}_{\mu\nu\sigma\lambda}(x,y,z)$$

Same kernel as for connected contributions, can be re-used ⇒ computational costs by the kernel are sub-dominant

2+2 graphs



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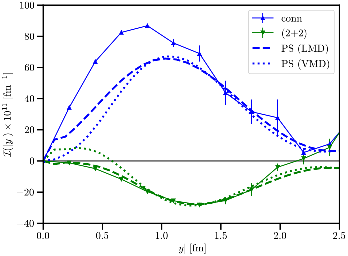
$$a_\mu^{(2+2),2} = \frac{m_\mu e^6}{3} 2\pi^2 \sum_{|y|} |y|^3 \sum_{x,z} \mathcal{L}^{(\mathrm{sym})}_{[\rho,\sigma];\mu\nu\lambda}(x,y)\ (y_\rho - 3z_\rho)\ \widetilde{\Pi}^{(2+2),(2)}_{\mu\nu\sigma\lambda}(x,y,z)$$

Take average as final result (⇒ improve statistics):

$$a_\mu^{(2+2)} = \frac{1}{2}\left(a_\mu^{(2+2),1} + a_\mu^{(2+2),2}\right)$$

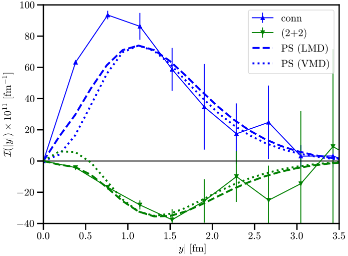
Light quarks: Integrand

PRELIMINARY



Integrand, $L = 3$ fm, $a = 0.1097$ fm

PRELIMINARY



Integrand, $L = 6$ fm, $a = 0.1097$ fm

- Very precise data for small y, more noise for larger y.
- Long distance dominated by light pseudoscalar exchange.
- Use information from pseudoscalar transition form factors (TFFs) to improve results.

Pseudoscalar pole contribution

- Use pseudoscalar TFF results [BMW '23] in order to supplement the integrand data at large y, where the signal is noise + finite size effect estimate/correction.
- Evaluate contributions for $P = \pi^0, \eta, \eta'$ to the 4pt-function numerically using transition form factor (TFF) data (assume LMD parameterization, data from lattice calculation [RBC-UKQCD '23]:

$$\Pi^P_{\mu\nu\sigma\lambda}(x,y,z) = \int_{uv} D_P(u-v)\left(\widetilde{M}^P_{\mu\nu}(u,x,y)\widetilde{M}^P_{\sigma\lambda}(v,z,0) + \cdots\right)$$

$$\widetilde{M}^P_{\mu\nu}(u,x,y) := i\epsilon_{\mu\nu\rho\sigma}\partial^{(x)}_\rho\partial^{(y)}_\sigma \int_{qk} F_{P\gamma^*\gamma^*}(q^2,k^2)\; e^{iq(x-u)}e^{iq(y-u)}$$

- Perform integration over position space variables numerically in finite volume.
- Use LMD (VMD) parameterization to get an upper (lower) bound for the pseudoscalar pole contribution
- π-pole contribution is enhanced in (separately considered) connected and 2+2 contributions:
  - $\mathcal{I}^{\mathrm{conn}} \approx \frac{34}{9}\mathcal{I}^\pi$
  - $\mathcal{I}^{(2+2)} \approx \mathcal{I}^\eta + \mathcal{I}^{\eta'} - \frac{25}{9}\mathcal{I}^\pi$

Light quarks: Integrand

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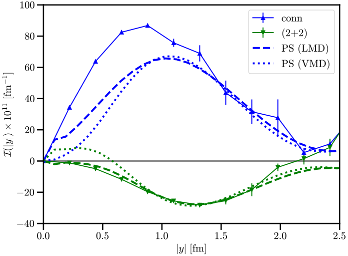

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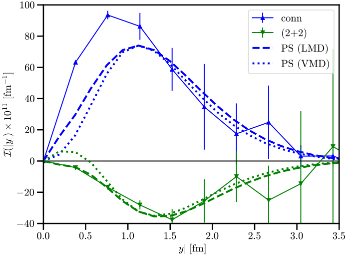
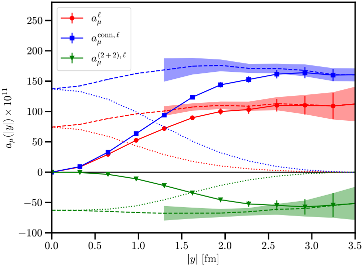


Integrand, $L = 6$ fm, $a = 0.1097$ fm

- Light pseudoscalar pole approximation in finite volume describes data well for $|y| > 1.5$ fm.
- Works for both volumes ⇒ Finite size effects are well described by pseudoscalar poles ⇒ Can be used for infinite volume extrapolation.
- Only small differences between LMD and VMD parameterization at large $|y|$.

Light quarks: Strategies

PRELIMINARY



Accumulative sum, $L = 6$ fm, $a = 0.0939$ fm

- Replace lattice data by pseudoscalar pole integrand once its contribution is small compared to the statistical error
- FSE correction: $v^P = a_\mu^{P,\mathrm{cont}} - a_\mu^P(a, V)$

Two approaches:

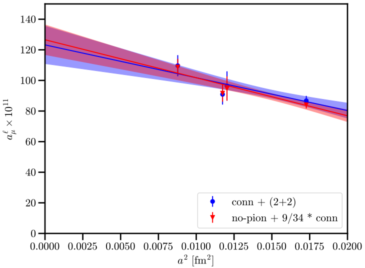
- Direct calculation of $\mathcal{I}^{\mathrm{cont}} + \mathcal{I}^{(2+2)}$:

$$a_\mu^\ell = \int_0^{y_{\mathrm{cut}}} \mathrm{d}y \left[\mathcal{I}^{\mathrm{cont}}(y) + \mathcal{I}^{(2+2)}(y)\right] + \int_{y_{\mathrm{cut}}}^{\infty} \mathrm{d}y \left[\mathcal{I}^{\pi}(y) + \mathcal{I}^{\eta}(y) + \mathcal{I}^{\eta'}(y)\right] + v^\pi + v^\eta + v^{\eta'}$$

- Calculate $a_\mu^{no-\pi} = \frac{25}{34} a_\mu^{\mathrm{conn}} + a_\mu^{(2+2)}$ (converges at $|y| \sim 2$ fm, smaller stat. error), add $\frac{9}{34} a_\mu^{\mathrm{conn}}$ (smaller stat. error) *[RBC/UKQCD '23]*.

Light quarks: continuum extrapolation

PRELIMINARY

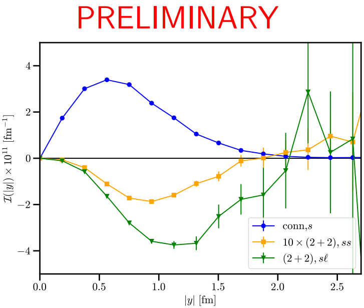


- 3 lattice spacings based on large volume only ($L = 6$ fm).
- Continuum extrapolation:

$$a_\mu^\ell(a^2) = a_\mu^\ell + \beta_2(\Lambda a)^2$$

- The two approaches agree perfectly.
- Continuum value (conn+(2+2)): $a_\mu^\ell = 123(12)(3)[13] \times 10^{-11}$ [PRELIMINARY]
- Compatible within 1σ with *[RBC/UKQCD '23]* and *[Mainz '21]*.

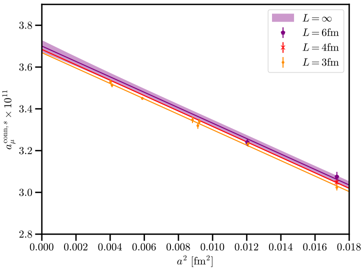
Strange connected: integrand



- Sub-percent precision, systematic errors become relevant.
- Have to take into account the error of the lattice spacing (need muon mass in lattice units for the kernel calculation).
- For the sea quark masses, have $m_s \neq m_s^{\text{phys}} \Rightarrow$ correction necessary.
- Finite volume effects are small but statistically relevant.
- Strategy: 11 ensembles with 3 volumes and 5 lattice spacings.

Strange connected: continuum extrapolation

PRELIMINARY



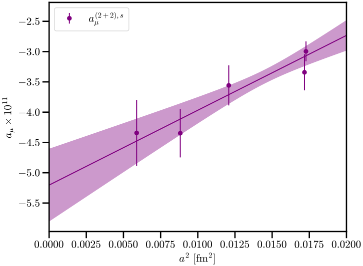
- Close to linear behavior in a^2, also consider quadratic term and scale dependence.
- Consider finite volume effects.
- Mismatch of strange quark mass.
- Ansatz:

$$a_\mu(a^2, M_{ss}) = a_\mu^{\mathrm{conn},s} + \beta_2(\Lambda a)^2 + \delta_2\ (\Lambda a)^2\ \alpha^n(a^{-1}) + \beta_4(\Lambda a)^4 + \gamma\frac{\delta M_{ss}}{M_{ss}^{\mathrm{phys}}} + \lambda\ e^{-Lm_\pi}$$

- Continuum value: $a_\mu^{\mathrm{conn},s} = 3.694(21)(11)[23]\times 10^{-11}$ [PRELIMINARY]
- RBC/UKQCD: $3.5(0.1)\times 10^{-11}$ *[RBC-UKQCD '23]*

Strange 2+2

PRELIMINARY



- Statistical error larger than for connected diagrams.
- Linear continuum extrapolation:

$$a_\mu^{(2+2),s/c}(a^2) = a_\mu^{(2+2),s/c} + \beta_2(\Lambda a)^2$$

- Continuum value: $a_\mu^{(2+2),s} = -5.19(59) \times 10^{-11}$ [PRELIMINARY] (large cancelation with connected contribution)
- **Total strange contribution**:
  - Our result: $a_\mu^s = -1.49(59)(1)[59] \times 10^{-11}$ [PRELIMINARY]
  - RBC/UKQCD: $-0.0(2.3) \times 10^{-11}$ *[RBC-UKQCD '23]*
  - Mainz: $-0.6(2.0) \times 10^{-11}$ *[Mainz '19]*

Content

Summary

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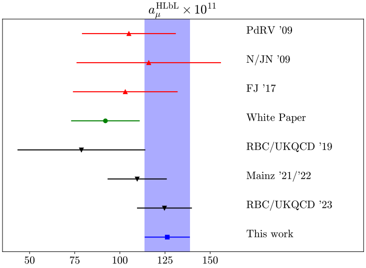
contribution	$a_\mu \times 10^{11}$
Light total	123(12)(3)[13]
Strange total	-1.61(66)(13)[68]
Charm total	3.690(53)(50)[73]
Sub-leading disc.	0.83(23)
Total	126(12)(3)[13]

PRELIMINARY



Achieved:

- Results for leading diagrams for light, strange and charm quark contributions
- Overall error around 10% and dominated by statistical error.

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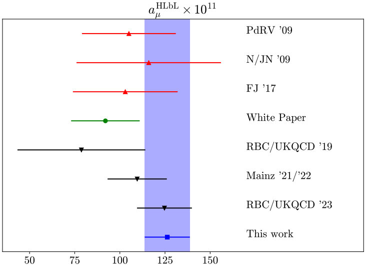

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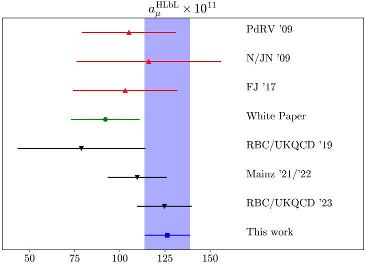

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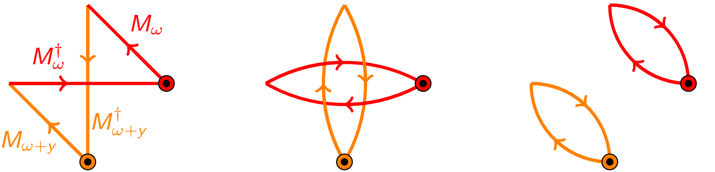
Thanks for your attention!

Backup slides

Technical approach

Conserved vector currents for staggered fermions $\chi(x)$, $\bar{\chi}(x)$:

$$j_\mu(x) = -\frac{1}{2}\eta_\mu(x)\left[\bar{\chi}(x+\hat{\mu})\ U^\dagger_\mu(x)\ \chi(x) + \bar{\chi}(x)\ U_\mu(x)\ \chi(x+\hat{\mu})\right]$$



- For given direction $\hat{n}$, consider all combinations of (y,ω) with $y = |y|\hat{n}$, $\omega = |\omega|\hat{n}$, exploit (anti-)periodicity.
- Repeat calculation for all $y' = y + h$, where $h_\mu = 0, \pm 1$ ($\mathcal{O}(a^2)$-smearing)
- Use local stochastic sources in color space $\Rightarrow$ Reduce number of matrix-matrix multiplications

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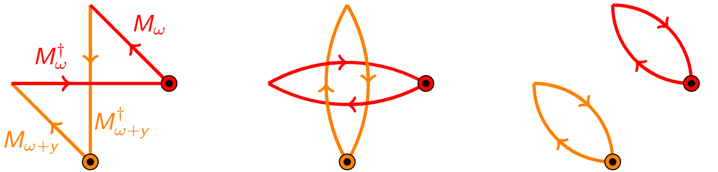

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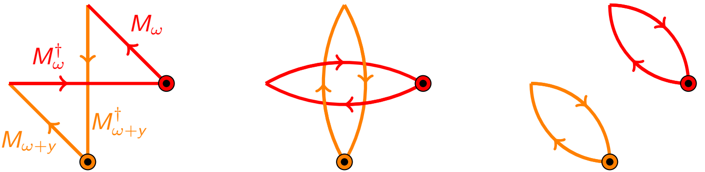
Technical approach

Conserved vector currents for staggered fermions $\chi(x)$, $\bar{\chi}(x)$:

$$j_\mu(x) = -\frac{1}{2}\eta_\mu(x)\left[\bar{\chi}(x+\hat{\mu})\ U_\mu^\dagger(x)\ \chi(x) + \bar{\chi}(x)\ U_\mu(x)\ \chi(x+\hat{\mu})\right]$$



- For given direction $\hat{n}$, consider all combinations of (y, ω) with $y = |y|\hat{n}$, $\omega = |\omega|\hat{n}$, exploit (anti-)periodicity.
- Repeat calculation for all $y' = y + h$, where $h_\mu = 0, \pm 1$ ($\mathcal{O}(a^2)$-smearing)
- Use local stochastic sources in color space $\Rightarrow$ Reduce number of matrix-matrix multiplications

Dealing with staggered tastes

$$a_\mu^{\mathrm{HLbL}} \propto \sum_{|y|} \mathcal{I}(|y|)$$

- Have to sample $\mathcal{I}(|y|)$ for a given set of points $|y|$
- In contrast to other discretizations (e.g. Wilson fermions), we have contributions for up to 16 tastes for a given y.
- Have to project onto the desired contribution

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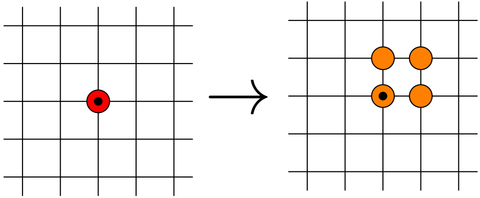
Project by applying "smearing":

$$\mathcal{I}(y) \to \mathcal{I}^{(1)}(y) = \prod_\mu \mathcal{S}_\mu^{(1)} \mathcal{I}(y) \qquad \mathcal{S}_\mu^{(1)} f(y) := \frac{f(y) + f(y+\hat{\mu})}{2}$$



staggered oscillations $\qquad$ $\mathcal{O}(a)$

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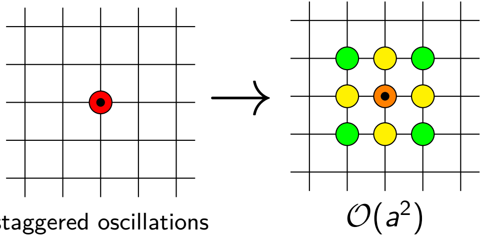
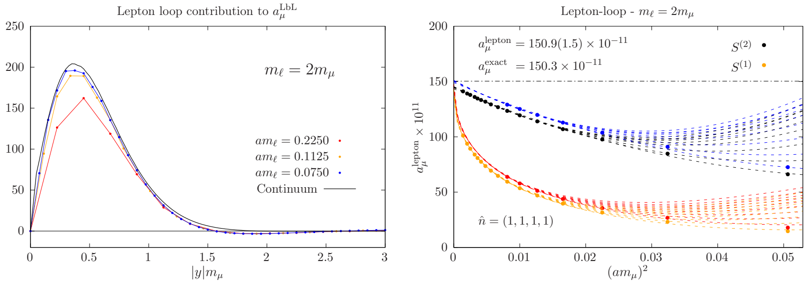
Project by applying "smearing":

$$\mathcal{I}(y) \to \mathcal{I}^{(2)}(y) = \prod_\mu \mathcal{S}_\mu^{(2)} \mathcal{I}(y) \qquad \mathcal{S}_\mu^{(2)} f(y) := \frac{f(y-\hat{\mu}) + 2f(y) + f(y+\hat{\mu})}{4}$$



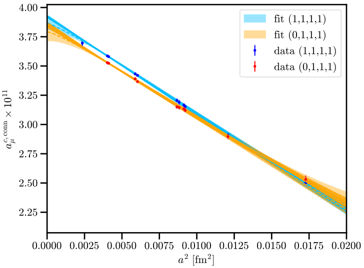
Test: Lepton loop on the lattice



- No staggered oscillations visible at the integrand level
- Ansatz: $a_\mu(a) = c_0 + c_1 a + c_2 a^2 + c_4 a^4$
- $a_\mu^{\rm lepton} = 150.9(1.5) \times 10^{-11}$ (lattice) vs $a_\mu^{\rm lepton} = 150.3 \times 10^{-11}$ (exact)
- **Values consistent** $\Rightarrow$ **First test successful**

Charm connected

PRELIMINARY



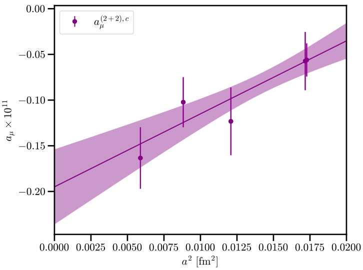
- Discretization artifacts: Sample in different directions regarding y
- Statistical errors small compared to systematic uncertainties.
- No visible finite size effects
- Charm quark mass fine tuning
- Ansatz:

$$a_\mu(a, M_{\eta_c}) = a_\mu^{\mathrm{conn},c} + \beta_2(\Lambda a)^2 + \delta_2(\Lambda a)^2\,\alpha_s^n(a^{-1}) + \beta_4(\Lambda a)^4 + \gamma\frac{\delta M_{\eta_c}}{M_{\eta_c}^{\mathrm{phys}}}$$

- Continuum value: $a^{\mathrm{conn},c}\mu = 3.889(7)(35)[36] \times 10^{-11}$ [PRELIMINARY]

Charm, 2+2

PRELIMINARY



- Linear continuum extrapolation:

$$a_\mu^{(2+2),s/c}(a^2) = a_\mu^{(2+2),s/c} + \beta_2(\Lambda a)^2$$

- Continuum values:
  - $a_\mu^{(2+2),c} = -0.193(40)$ [PRELIMINARY]