Progress on the Hadronic Vacuum Polarizaiton Contribution to Muon $g-2$ from Lattice QCD

Vaishakhi Moningi
University of Connecticut

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Collaborators

HVP with staggered fermions
Thomas Blum (UConn), Luchang Jin (UConn),Christopher Aubin (Fordham), Maarten Golterman (SFSU), Santiago Peris (Barcelona)

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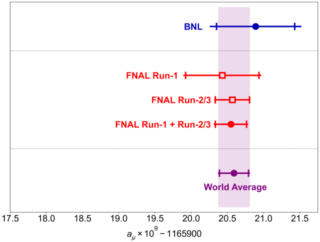
Muon g-2 experimental measurement

- Anomalous magnetic moment of the muon $a_\mu = (g-2)/2$
- Fermilab results stand at 0.203 ppm!

Run-2/3 Result: FNAL + BNL Combination

a_μ(FNAL) = 0.00 116 592 055(24) [203 ppb]



- FNAL combination: **203 ppb** uncertainty
- Both FNAL and BNL dominated by statistical error
- Combined world average **dominated by FNAL** values.

a_μ(Exp) = 0.00 116 592 059(22) [190 ppb]

Fermilab

Theoretical framework

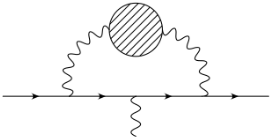


Figure 1: The quark connected diagram contributing to the HVP

$$a_\mu^{\mathrm{HVP}} = 4\alpha^2 \int_0^\infty dq^2 f\left(q^2\right) \hat{\Pi}\left(q^2\right) \xrightarrow[\text{[Bernecker and Meyer 2011]}]{\text{time-mom rep}} 2\sum_{t=0}^{T/2} w(t)C(t) \quad (1)$$

where $\hat{\Pi}\left(q^2\right) = \Pi\left(q^2\right) - \Pi(0)$ is the subtracted HVP

$$f(q^2) = \frac{m_\mu^2 q^2 Z^3 (1 - q^2 Z)}{1 + m_\mu^2 q^2 Z^2},$$

$$Z = -\frac{q^2 - \sqrt{q^4 + 4m_\mu^2 q^2}}{2m_\mu^2 q^2}.$$

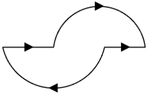
$$w(t) = 4\alpha^2 \int_0^\infty dq^2 f(q^2) \left[\frac{\cos qt - 1}{q^2} + \frac{t^2}{2}\right]$$

Two-point correlation function

- 2-point current-current function

$$C(t) = \frac{1}{3}\sum_{\vec{x},i}\langle J^i(\vec{x},t)J^i(0)\rangle = \frac{1}{3}\sum_{\vec{x}}\left\langle M^{-1}_{0,x}\gamma_\mu M^{-1}_{x,0}\gamma_\nu\right\rangle \qquad (2)$$

- Here, $J_\mu(x) = [\bar{\psi}\gamma_\mu\psi](x)$ is the electromagnetic current.
- On the lattice this current is not conserved, so we use a point-split current that is exactly conserved,



$$J^\mu(x) = \frac{1}{2}\eta_\mu(x)(\bar{\chi}(x+\hat{\mu})U^\dagger_\mu(x)\chi(x) + \bar{\chi}(x)U_\mu(x)\chi(x+\hat{\mu}))$$

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Motivation

From the spectral decomposition of the propagator

$$S(x,y) = \sum_{\lambda \leq \lambda_{low}} \frac{\langle x|\lambda\rangle \langle \lambda|y\rangle}{\lambda} + \sum_{\lambda > \lambda_{low}} \frac{\langle x|\lambda\rangle \langle \lambda|y\rangle}{\lambda} = S_L + S_H$$

we separate C(t) into four parts: low-low, low-high, high-low, and high-high[Giusti et al. 2004].

$$C_{\mu\nu}(t) = \sum_{x,y} \mathrm{Tr}\, \gamma_\mu G(x,y) \gamma_\mu G(y,x) = C_{LL} + C_{LH} + C_{HL} + C_{HH} \qquad (3)$$

In the previous work [Aubin et al. 2020; Aubin et al. 2022], C(t) was just divided into pure low-mode and the rest.

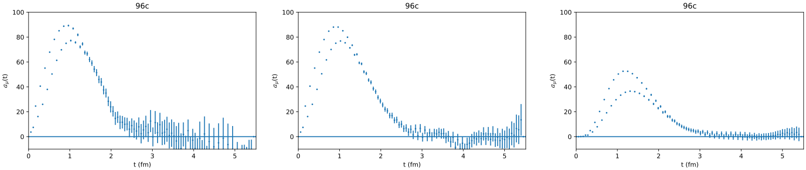


Figure 2: The integrand $(w(t)C(t))$ in Eq. (1). No LMA (left), total (middle), LMA only (right).

- In previous work, we used LMA for noisy long-distance part of the correlator.
- LL part yields full-volume average for both source and sink points.
- The rest (HL+LH+HH) averaged over small number of source points.
- LL part has smaller fluctuations compared to the total (Fig. 2).

Algorithm improvements: High-Low contribution

- Now, we would like to compute HL part separately instead of together with the HH[1].
- The high-low contribution is

$$C_{HL} = \sum_{n} \sum_{\boldsymbol{y}} \langle n|y\rangle U_{\nu}(y) G_{H}(y+\hat{\nu}, x+\hat{\mu}) U_{\mu}^{\dagger}(x) \frac{\langle x|n\rangle}{\lambda_n} + 3 \text{ other terms} \quad (4)$$

[1]We thank Simon Kuberski and the Mainz group for discussions

Combining eigenvectors with random coefficients

The calculation of the low-high part is still expensive ($N_T \times N_{\text{low}}$ sources). To dramatically reduce the cost,

- we combine low-mode sources on a time-slice using unique random numbers for each mode.
- Contract at the sink with the same random numbers to eliminate unwanted cross-terms on average.

$$\left(\frac{r_0}{\sqrt{\lambda_0}}\langle 0| + \frac{r_1}{\sqrt{\lambda_1}}\langle 1|\right)\left(\frac{r_0}{\sqrt{\lambda_0}}|0\rangle + \frac{r_1}{\sqrt{\lambda_1}}|1\rangle\right) = \frac{r_0^2}{\lambda_0}\langle 0|0\rangle + \frac{r_1^2}{\lambda_1}\langle 1|1\rangle + \frac{r_0 r_1}{\sqrt{\lambda_0\lambda_1}}\underbrace{(\langle 0|1\rangle + \langle 1|0\rangle)}_{\textit{crossterms}}$$

$$= C_{\text{exact}} \quad (\because r_i^2 = 1,\ \langle r_i r_j\rangle = \delta_{ij})$$

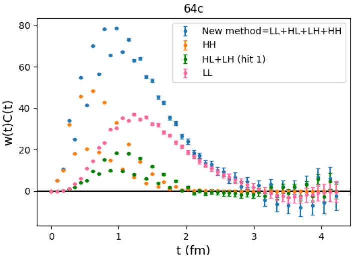
λ_i = eigenvalues of Dirac operator

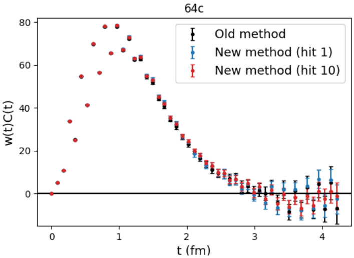
- This adds random noise which can be reduced by doing more "hits" with additional random-sources.

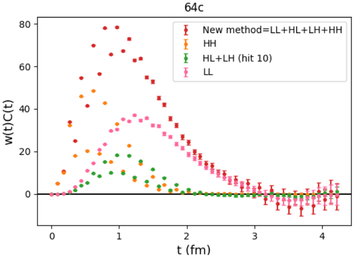
Results

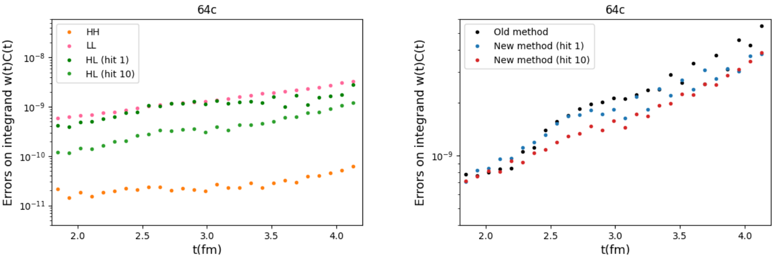
m_π (MeV)	a (fm)	size	L	configs (LL-HL-HH)
130	0.087	$64^3 \times 96$	5.62	31-31-31
134	0.042	$144^3 \times 288$	6.048	6-18-27

- Reduced $N_T \times N_{\rm low}$ solves to $N_T \times N_{\rm hits}$ solves
- 8000 low modes were used.









- Error from high-low part error is suppressed from low-low errors.
- Our current method with just "1 hit" shows an improvement of 8% and with "10 hits" 17.3% in the long distance window (2.6-3.4 fm)
- Low-low part error dominates in the total!

Algorithm improvements: Low-Low contribution

- The low-low part of the correlation function is

$$C_{LL} = \sum_{m,n} \sum_{\vec{x}} \frac{1}{\lambda_m \lambda_n} \Lambda^{\dagger}_{\mu}(x)_{mn} \Lambda^{\dagger}_{\nu}(y)_{nm} + \cdots \quad (5)$$

the meson field defined as

$$(\Lambda_{\mu}(t))_{n,m} = \sum_{\vec{x}} \langle n|x\rangle \, U_{\mu}(x) \, \langle x+\mu|m\rangle$$

- This scales linearly in the size of the eigenvectors and quadratically with the number of eigenvectors ($N_S^3 \times N_T$)
- To have a significant speedup, we "sparsen" the eigenvectors.
- As a full volume average may be wasteful as nearby points will be almost 100% correlated on a fine lattice.
- We sparsen randomly to choose the location for the hypercube on a timeslice.

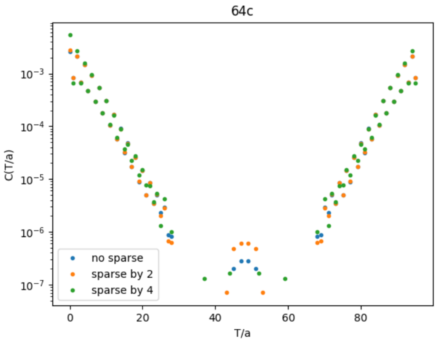
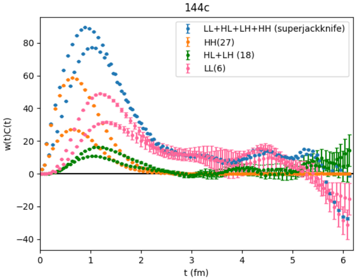
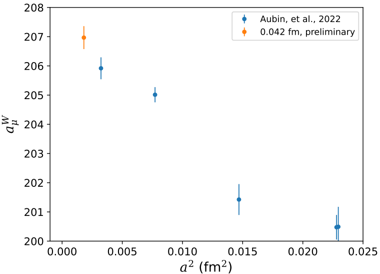


Figure 3: Low-low contribution from contracting the meson fields

- For the demonstration purpose, we have used 800 low modes on 1-configuration.
- Sparsening by (s, t) reduces the number of eigenvectors required to compute our meson fields from $N_S^3 \times N_T$ to $(N_S/s)^3 \times (N_T/t)$.

Preliminary results on $144^3 \times 288$ ensemble





$a_\mu \times 10^{10}$	window (t_0, t_1, Δ) (fm)
207.24(34)	(0.4, 1.0, 0.15)
94.57(88)	(1.5,1.9,0.15)

- Intermediate window quantity:

$$a_\mu = 2 \sum_{t=0}^{T/2} C(t) w(t) \left(\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)\right)$$

with, $\Theta(t, t', \Delta) = \frac{1}{2}\left(1 + \tanh \frac{t-t'}{\Delta}\right)$

- Errors shown are Statistical

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Summary

- The new method reduces the statistical noise in the long-distance part of the two-point correlation function.
- The added expense of separately computing HL is a trade-off for improving the errors in this region and using fewer sources for the HH part (which now doesn't have the extra noise of the HL contributions).
- Physical point calculations nearly complete at a $= 0.087$ fm.
- And $a = 0.042$ fm calculations are in progress.

Acknowledgements

- Highly-improved staggered quark (HISQ) ensemble from the MILC collaboration with the HPQCD value $w_0 = 0.1715(9)$ fm

$\approx a$/fm	L/fm	$N_s^3 \times N_t$	$am_l^{\text{sea}}/am_s^{\text{sea}}/am_c^{\text{sea}}$	w_0/a
0.15	4.85	$32^3 \times 48$	0.002426/0.0673/0.8447	1.13227(18)
0.12	5.81	$48^3 \times 64$	0.001907/0.05252/0.6382	1.41060(28)
0.09	5.61	$64^3 \times 96$	0.00120/0.0363/0.432	1.95148(41)
$0.09^\star$	5.61	$64^3 \times 96$	0.001326/0.03636/0.4313	1.95021(57)
0.06	5.45	$96^3 \times 128$	0.0008/0.022/0.260	3.01838(92)
0.04	6.12	$144^3 \times 288$	0.000569/0.01 555/0.1827	4.03242(195)

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