Symplectic Quantization

A new deterministic approach to the dynamics of quantum fluctuations inspired by statistical mechanics

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Symplectic Quantization

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Outline

- Conceptual introduction
- 1 From Path integrals to Stochastic quantization: stochastic dynamics for quantum fluctuations
- 2 Symplectic quantization: Hamiltonian dynamics for quantum fluctuations
- 3 Proof of equivalence
- 4 First numerical simulations: the scalar field
- 5 Conclusions and perspectives

References

'Perturbation theory without gauge fixing', G. Parisi, Y.S. Wu, *Sci. Sin.* **24**, 483 (1981).

'Symplectic quantization I: dynamics of quantum fluctiations in a relativistic field theory', G. Gradenigo, R. Livi, Foundation of Physics 51 (2021).

Symplectic quantization and the Feynman propagator: a new real-time numerical approach to lattice field theory

M.Giachello, G. Gradenigo e-Print: 2403.17149 [hep-lat]

1 FROM PATH INTEGRALS TO STOCHASTIC QUANTIZATION

$$\mathcal{Z}(\hbar) = \int \mathcal{D}\phi \; e^{\frac{i}{\hbar} S[\phi]}$$

Feynman Path Integral

$$dx_0 = -\, i d\tilde{x}_0$$

Wick rotation

$$\mathcal{Z}_E(\hbar) = \int \mathcal{D}\phi \; e^{-\frac{1}{\hbar} S_E[\phi]}$$

Euclidean Path Integral

SIGN PROBLEM

In the Euclidean space it is possible to give to the path integral a probabilistic interpretation

$$P_\hbar[\phi] = \exp(-S_E[\phi]/\hbar)/\mathcal{Z}_E(\hbar)$$

THIS WAS NOT POSSIBLE IN THE ORIGINAL MINKOWSKIAN METRIC

BOLTZMANN WEIGTH: we need to sample configurations according to this probability distribution

POSSIBLE METHODS IN LITERATURE:

1. Monte Carlo methods of importance
2. Parisi-Wu's Stochastic Quantisation

Dynamics in QFT: STOCHASTIC QUANTIZATION

2. Parisi-Wu's Stochastic Quantisation

Stochastic algorithm to update the field configuration in space-time

$$P[\phi_1(x)|\phi_0(x)] \sim e^{-(S_E[\phi_1]-S_E[\phi_0])/\hbar}$$

$$\Phi_n[\phi_0(x)] = \phi_n(x)$$

Stochastic dynamics

$$\phi(x) : \mathbb{R}^4 \longrightarrow \mathbb{R}$$

Sequence of configurations parametrised by an integer index: discrete 'time' of the algorithm

$$\underbrace{\{\phi_1(x), \phi_2(x), \phi_3(x), \ldots, \phi_N(x)\}}$$

$$\phi_i(x) : \mathbb{R}^4 \times \mathbb{Z} \longrightarrow \mathbb{R}$$

DYNAMICS: ordered sequence of field configurations

TIME: labelling of the ordered sequence

Dynamics in QFT: STOCHASTIC QUANTIZATION

2. Parisi-Wu's Stochastic Quantization

We promote the algorithmic time to a continuous parameter → We model the sequence of quantum fluctuations by means of a Langevin equation

$$\phi(x,\tau) : \mathbb{R}^4 \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$\boxed{\frac{\partial}{\partial \tau}\phi(x,\tau) = -\frac{\delta S_E[\phi]}{\delta \phi(x)} + \eta(x,\tau)}$$

$$\langle \eta(x,\tau)\eta(y,\tau')\rangle = 2\hbar\, \delta^{(4)}(x-y)\, \delta(\tau-\tau')$$

$$\langle \eta(x,\tau)\rangle = 0$$

The Langevin equation is equivalent to Fokker-Plank

$$\frac{d}{dt}P_t[\phi] = \hat{H}_{FP} \circ P_t[\phi]$$

$$P_{eq}[\phi] \propto e^{-\frac{1}{\hbar}S_E[\phi]}$$

From a stochastic to a deterministic dynamics: the definition of a conserved quantity

1. Analogously to what done in Hybrid Monte Carlo (HMC) , we can interpret the stochastic dynamics as the approximation of a deterministic one.

A. $$\frac{\partial}{\partial \tau}\phi(x,\tau) = -\frac{\delta S_E[\phi]}{\delta \phi(x)} + \eta(x,\tau)$$

B. $$\frac{d^2\phi}{d\tau^2} = -\frac{\delta S_E[\phi]}{\delta \phi(x,\tau)}$$

2. We introduce a conjugate momentum (with respect to the intrinsic time)

$$\pi(x,\tau) \propto \dot{\phi}(x,\tau)$$

3. We define in a reasonable way the Hamiltonian of our system

$$\boxed{\mathcal{A}_E(\phi,\pi) = \frac{c_s^2}{2}\int d^4x\ \pi^2(x) + S_E[\phi]}$$

$$S_E[\phi] = \int d^4x \left[\sum_{\mu=0}^{d-1}\left(\frac{\partial \phi}{\partial x_\mu}\right)^2 + V(\phi)\right]$$

The relativistic action is taken as potential for a generalised action

SYMPLECTIC QUANTIZATION: HAMILTONIAN DYNAMICS

From a stochastic to a deterministic dynamics: equations of motion

STOCHASTIC DYNAMICS

$$\frac{\partial}{\partial \tau}\phi(x,\tau) = -\frac{\delta S_E[\phi]}{\delta \phi(x)} + \eta(x,\tau)$$

$\tau =$ ALGORITHMIC TIME

HAMILTONIAN DYNAMICS

$$\dot{\pi}(x,\tau) = -\frac{\delta \mathcal{A}_E}{\delta \phi(x,\tau)}$$

$$\dot{\phi}(x,\tau) = \frac{\delta \mathcal{A}_E}{\delta \pi(x,\tau)}$$

$\tau =$ INTRINSIC TIME

The physical ‘microscopic dynamics’ of quantum fluctuations

Probabilistic interpretation and well posedness

HAMILTONIAN DYNAMICS

$$\dot{\pi}(x,\tau) = -\frac{\delta \mathcal{A}_E}{\delta \phi(x,\tau)}$$

$$\dot{\phi}(x,\tau) = \frac{\delta \mathcal{A}_E}{\delta \pi(x,\tau)}$$

NO NEED OF IMAGINARY TIME

We can go back to the original metric

Functional Approach to Quantum Field Theory probabilistically well defined

Microcanonical Postulate

All field configurations characterised by identical value of ACTION are equally likely

$$P_A[\phi,\pi] = \frac{1}{\Omega_E(A)} \delta\left[A - \mathcal{A}_E(\phi,\pi)\right]$$ **PROBABILITY**

$$\Omega_E(A) = \int \mathcal{D}\phi \; \mathcal{D}\pi \; \delta\left[A - \mathcal{A}_E(\phi,\pi)\right]$$ **PARTITION SUM**

ERGODICITY ASSUMPTION

SYMPLECTIC QUANTIZATION: HAMILTONIAN DYNAMICS

Probabilistic interpretation and well posedness

NO NEED OF IMAGINARY TIME

Original Relativistic Action

$$S[\phi] = \int d^4x \; \mathcal{L}(\partial_\mu \phi, \phi) \qquad \mathcal{L}(\partial_\mu \phi, \phi) = \partial_\mu \phi \partial^\mu \phi - V(\phi)$$

Symplectic Quantisation Action

$$\mathcal{A}[\phi, \dot{\phi}] = \int d^4x \left[\frac{1}{2c_s^2} \dot{\phi}^2 - \mathcal{L}(\partial_\mu \phi, \phi) \right] = \int d^4x \; \mathcal{H}(\phi, \dot{\phi})$$

$$\dot{\pi}(x,\tau) = -\frac{\delta \mathcal{A}[\phi,\pi]}{\delta \phi(x,\tau)}$$

$$\dot{\phi}(x,\tau) = \frac{\delta \mathcal{A}[\phi,\pi]}{\delta \pi(x,\tau)}$$

$$\mathcal{A}[\phi,\pi] = \int d^4x \left[\frac{c_s^2}{2} \pi^2(x) - \left(\frac{\partial \phi}{\partial x_0} \right)^2 + \sum_{i=1}^{d-1} \left(\frac{\partial \phi}{\partial x_i} \right)^2 + V[\phi(x)] \right]$$

MICROCANONICAL ENSEMBLE

$$P_A[\phi,\pi] = \frac{1}{\Omega(A)} \delta\left[A - \mathcal{A}(\phi,\pi) \right]$$

PROBABILITY

$$\Omega(A) = \int \mathcal{D}\phi \; \mathcal{D}\pi \; \delta\left[A - \mathcal{A}(\phi,\pi) \right]$$

PARTITION SUM

Role of the intrinsic time

HAMILTONIAN DYNAMICS

$$\dot{\pi}(x,\tau) = -\frac{\delta \mathcal{A}[\phi,\pi]}{\delta \phi(x,\tau)}$$

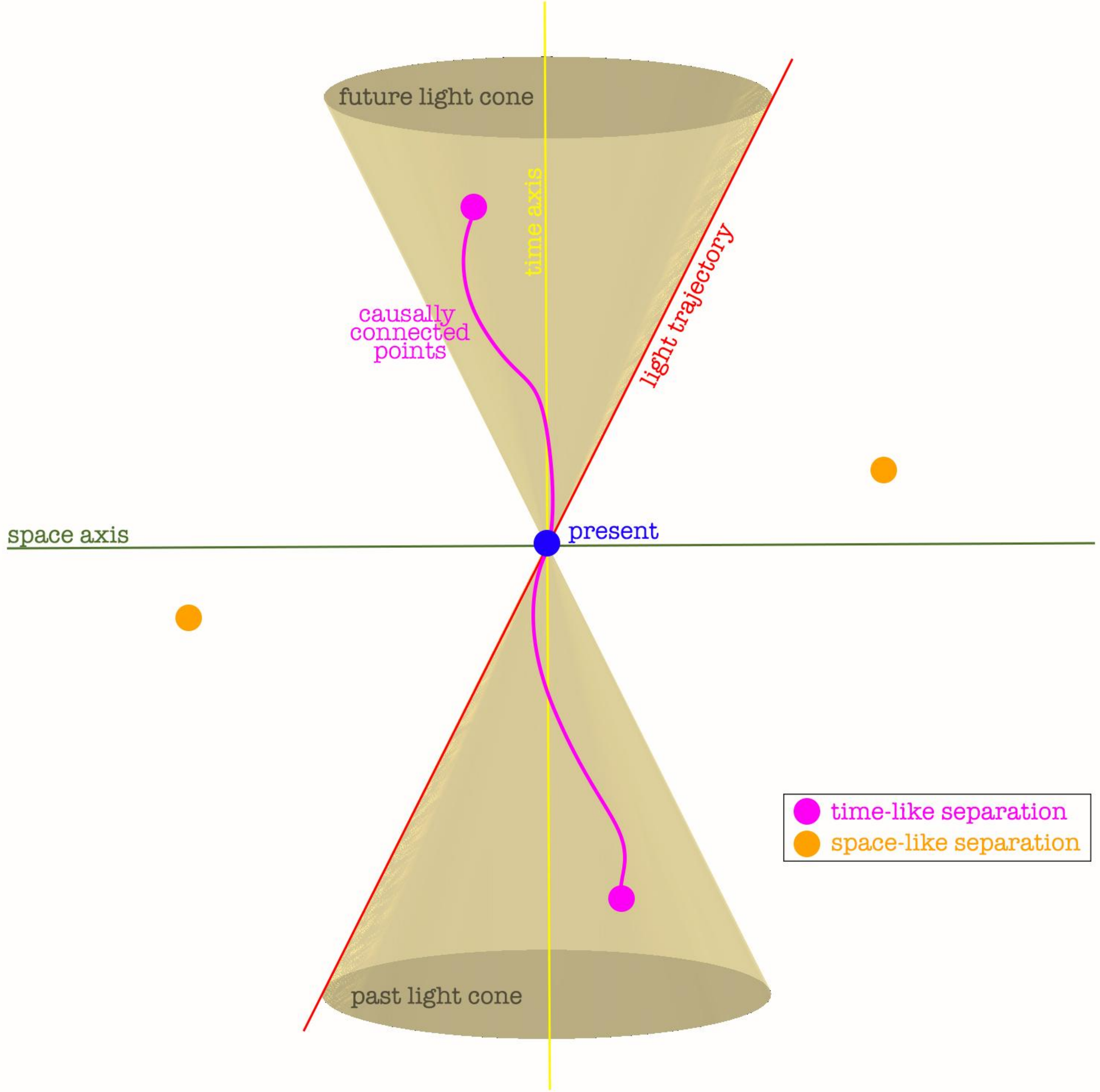
$$\dot{\phi}(x,\tau) = \frac{\delta \mathcal{A}[\phi,\pi]}{\delta \pi(x,\tau)}$$

$$\phi(x_0, \mathbf{x}, \tau)$$

Physical time as the fourth axis of Minkowski space-time

Parameter labelling the dynamics of quantum fluctuations of a given point of space-time (intrinsic time)

Recovering causality



Purely space-like separation

$$x - y = (0, \delta x_1, 0, 0)$$

Exponential damping

$$\langle \phi(x)\phi(y) \rangle \sim e^{-m\delta x_1}$$

Purely time-like separation

$$X - Y = (\delta x_0, 0, 0, 0)$$

Propagation

$$\langle \phi(x)\phi(y) \rangle \sim e^{-im\delta x_0}$$

Relation to Feynman Path Integral

This formalism is well defined because we can go back to the Feynman Path integral by means of a Fourier transform (change of statistical ensemble):

$$\mathcal{Z}(\hbar) = \int_{-\infty}^{\infty} dA \; e^{-iA/\hbar} \; \Omega(A)$$

$$\mathcal{Z}(\hbar) = \int dA \; e^{-iA/\hbar} \int \mathcal{D}\phi \mathcal{D}\pi \; \delta\left(A - \mathcal{A}(\phi,\pi)\right)$$

$$\mathcal{Z}(\hbar) = \int \mathcal{D}\phi \mathcal{D}\pi \;\; \exp\left\{ -\frac{i}{\hbar} \int d^4x \; \pi^2(x) + \frac{i}{\hbar} S[\phi] \right\}$$

Dependence on momenta is quadratic, the generalised action is separable

$$\mathcal{Z}(\hbar) \propto \int \mathcal{D}\phi \; e^{iS[\phi]/\hbar}$$

3 PERTURBATIVE EQUIVALENCE OF THE MICROCANONICAL GENERATING FUNCTIONAL WITH THE PATH INTEGRAL FORMULATION

A. Expand the field $\phi(x)$ in an orthonormal basis

$$\phi(x) = \sum_{n=1}^{N} \phi_n(x) c_n \qquad \int d^d x \, \phi_n(x)\phi_m(x) = \delta_{mn} \qquad \int \mathcal{D}_N \phi \equiv \prod_{n=1}^{N} \int_{-\infty}^{\infty} dc_n \qquad N = \frac{1}{\pi^d} V \Lambda^d$$

B. Prove that in the thermodynamic limit ($N \to \infty$) the microcanonical ensemble is equivalent to the canonical

$$\Omega[\mathcal{A}, J] = \frac{1}{\Omega(\mathcal{A}, 0)} \int \mathcal{D}\phi \mathcal{D}\pi \, \delta\left(\mathcal{A} - \frac{\pi^2}{2} + S[\phi] + \boxed{J\phi}\right)$$

$$\mathcal{Z}[\hbar, J] = \int_{-\infty}^{\infty} d\mathcal{A} \, e^{-\frac{i}{\hbar}\mathcal{A}} \, \Omega(\mathcal{A}, J) = \int_{-\infty}^{\infty} d\mathcal{A} \, e^{-\frac{i}{\hbar}\mathcal{A} + \ln \Omega[J, \mathcal{A}]}$$

C. We then use the SADDLE POINT APPROXIMATION on

$$\ln \mathcal{Z}[\hbar, J] = -\frac{i}{\hbar}\mathcal{A}(\hbar, J) + \ln \Omega[\mathcal{A}(\hbar, J), J] + \ln \int_{-\infty}^{\infty} d\mathcal{A} \, e^{\frac{\mathcal{A}^2}{2}\left(\frac{\partial^2}{\partial \mathcal{A}^2} \ln \Omega[\mathcal{A}, J]|_{\mathcal{A}=\mathcal{A}(\hbar, J)}\right) + \mathcal{O}(\mathcal{A}^3)}$$

$$\boxed{\left.\frac{\partial}{\partial \mathcal{A}} \ln \Omega[\mathcal{A}, J]\right|_{A = A(\hbar, J)} = \frac{i}{\hbar}}$$

PERTURBATIVE EQUIVALENCE OF THE MICROCANONICAL GENERATING FUNCTIONAL WITH THE PATH INTEGRAL FORMULATION

D. We consider the one loop expansion around the classical field

$$S[\phi] = S[\phi_c] + \frac{\delta S}{\delta \phi}\Big|_{\phi=\phi_c}(\phi - \phi_c) + \frac{1}{2}\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\Big|_{\phi=\phi_c}(\phi-\phi_c)^2 + \dots$$

$$\Omega[\mathcal{A}, J] = \frac{1}{\Omega[\mathcal{A}, 0]} \times \int \mathcal{D}\phi_N \mathcal{D}\pi_N \,\delta\left(A - \frac{\pi^2}{2} + S[\phi_c] + J\phi_c + \frac{1}{2}S''[\phi_c]\phi^2\right)$$

E. The integral is the area of a 2N dimensional sphere

$$\Omega[\mathcal{A}, J] = \mathrm{Det}^{-\frac{1}{2}}(-S''[\phi_c])\frac{1}{\Gamma(N)}(\mathcal{A} + S[\phi_c] + J\phi_c)^{N-1}$$

F. If we impose the saddle point equation and consider the leading order we find

$$\mathcal{A}(\hbar, J) = \mathcal{A}(\hbar, 0) = -i\hbar N$$

$$\begin{aligned}\Omega_T[\hbar, J] &= \lim_{N\to\infty}\Omega[\mathcal{A}(\hbar, J), J] \\ &= \lim_{N\to\infty}\mathrm{Det}^{-\frac{1}{2}}(-S''[\phi_c])\left(\frac{\mathcal{A}(\hbar, J)}{N}\right)^N \\ &\quad\times\left(1 + \frac{S[\phi_c] + J\phi_c}{\mathcal{A}(\hbar, J)}\right)^N \\ &= \mathrm{Det}^{-\frac{1}{2}}(-S''[\phi_c])e^{\frac{i}{\hbar}S[\phi_c] + \frac{i}{\hbar}J\phi_c}\end{aligned}$$

PERTURBATIVE EQUIVALENCE OF THE MICROCANONICAL GENERATING FUNCTIONAL WITH THE PATH INTEGRAL FORMULATION

G. We obtain the one-loop Minkowskian effective action of a scalar bosonic theory

$$\ln \Omega_T[\hbar, J] = -\frac{1}{2}\ln \mathrm{Det}(-S''[\phi_c]) + \frac{i}{\hbar}S[\phi_c] + \frac{i}{\hbar}J\phi_c$$

$$\mathcal{Z}[\hbar, J] = e^{-\frac{i}{\hbar}\mathcal{W}[\hbar, J]} \qquad -\frac{i}{\hbar}\mathcal{W}[\hbar, J] = \ln \Omega_T[\hbar, J]$$

$$\boxed{\mathcal{W}[\hbar, J] = -S[\phi_c] - J\phi_c - \frac{i\hbar}{2}\ln \mathrm{Det}(-S''[\phi_c])}$$

GENERATING FUNCTION OF CONNECTED CORRELATION

$$\boxed{\Gamma[\hbar, \phi_c] = -\mathcal{W}[\hbar, J] - J\phi_c = S[\phi_c] + \frac{i\hbar}{2}\ln \mathrm{Det}(-S''[\phi_c])}$$

EFFECTIVE ACTION

MICROCANONICAL QUANTUM FIELD THEORY (Andrew E. Strominger)

NUMERICAL SIMULATIONS

$\lambda\phi^4$ scalar field theory in 1+1 dimensions

$$\boxed{\begin{aligned}\dot{\pi}(x,\tau) &= -\frac{\delta\mathcal{A}[\phi,\pi]}{\delta\phi(x,\tau)}\\ \dot{\phi}(x,\tau) &= \frac{\delta\mathcal{A}[\phi,\pi]}{\delta\pi(x,\tau)}\end{aligned}}$$

$$S[\phi] = \frac{1}{2}\int d^2x\,\left\{(\partial_{x_0}\phi)^2 - (\partial_{x_1}\phi)^2 - m^2\phi^2 - \frac{1}{2}\lambda\phi^4\right\}$$

$$\boxed{\ddot{\phi}(x_{\mathbf{i}},\tau) = -\Delta_0\phi + \Delta_1\phi - m^2\phi(x_{\mathbf{i}},\tau) - \cancel{\lambda\phi^3(x_{\mathbf{i}},\tau)}}$$

We remove the interaction

FREE CASE

$\lambda = 0$

$$\ddot{\phi}(k,\tau) + \omega_k^2\phi(k,\tau) = 0 \qquad \omega_k^2 = -k_0^2 + k_1^2 + m^2$$

$$\phi(k,\tau) = \phi(k,0)\cos(\omega_k\tau) + \frac{\dot{\phi}(k,0)}{\omega_k}\sin(\omega_k\tau) \quad \omega_k^2 > 0$$

BLOW-UP SOLUTION!

$$\phi(k,\tau) = \phi(k,0)\cosh(z_k\tau) + \frac{\dot{\phi}(k,0)}{z_k}\sinh(z_k\tau) \quad \omega_k^2 < 0$$

$$iz_k = \sqrt{\omega_k^2}$$

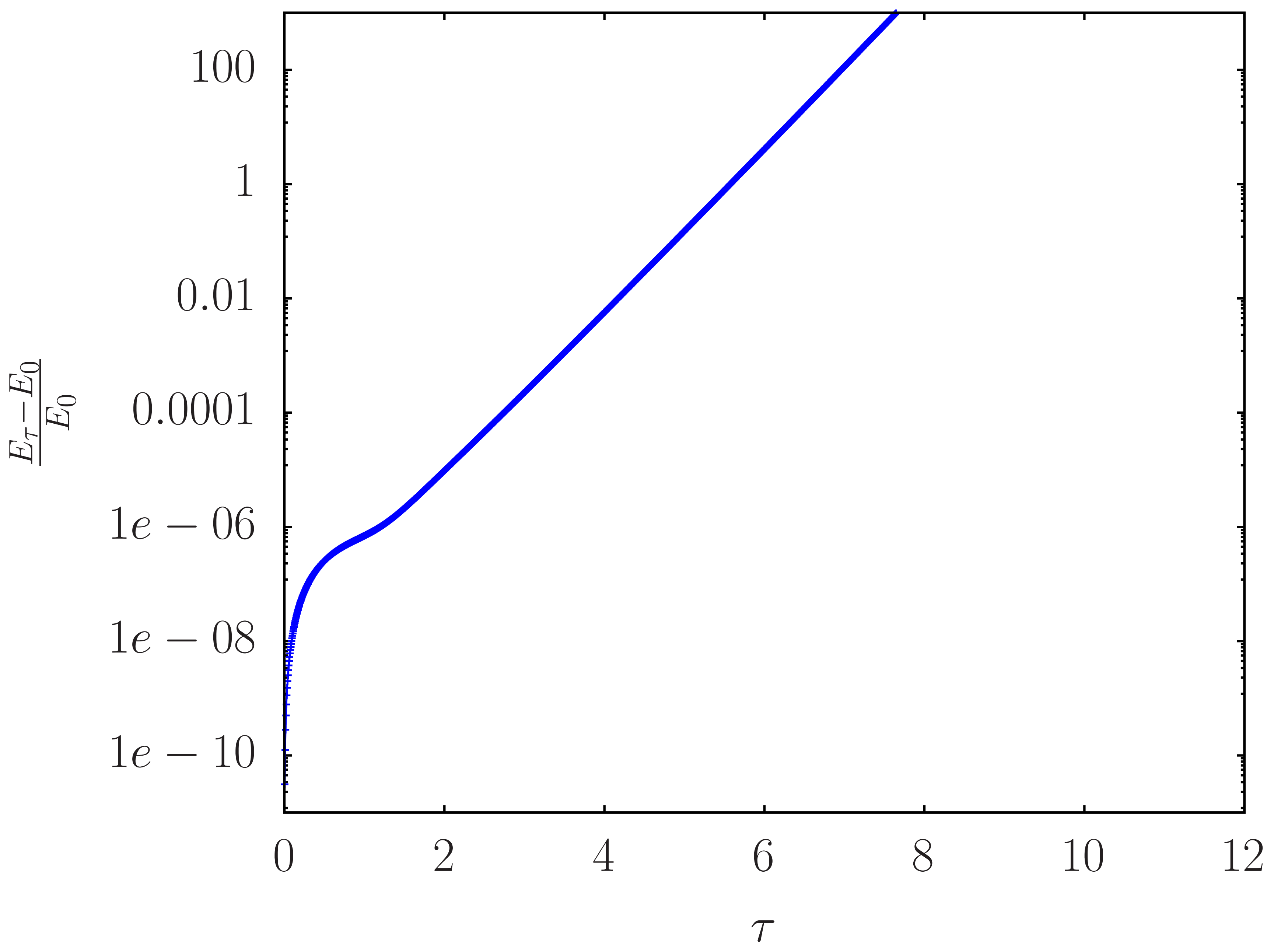
NUMERICAL SIMULATIONS: RESULTS

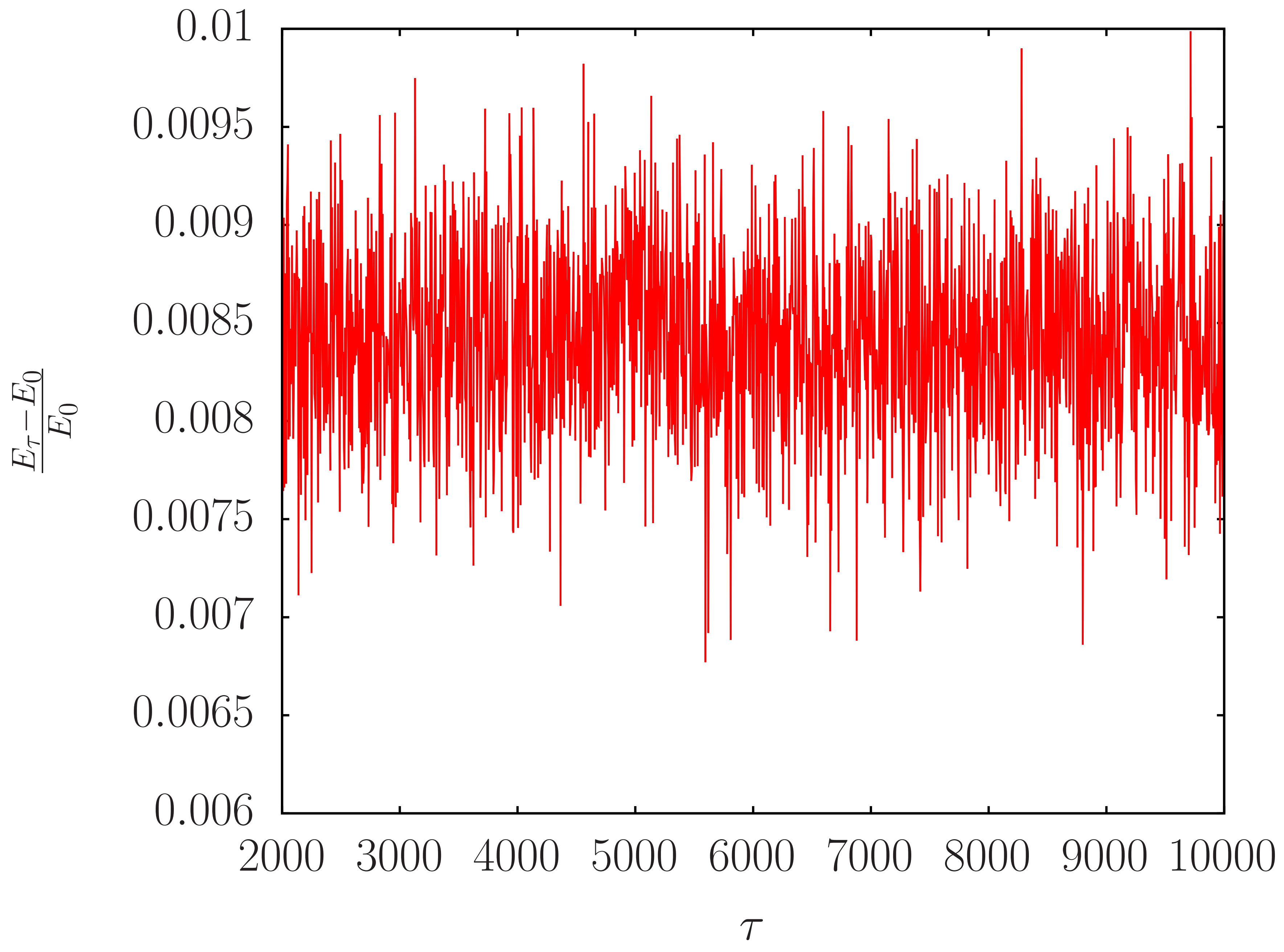
POSSIBLE SOLUTIONS TO THE BLOW UP:

The role of the interaction

$$\ddot{\phi}(x_{\mathbf{i}},\tau) = -\Delta_0\phi + \Delta_1\phi - m^2\phi(x_{\mathbf{i}},\tau)\boxed{-\lambda\phi^3(x_{\mathbf{i}},\tau)}$$

RESTORING FORCE



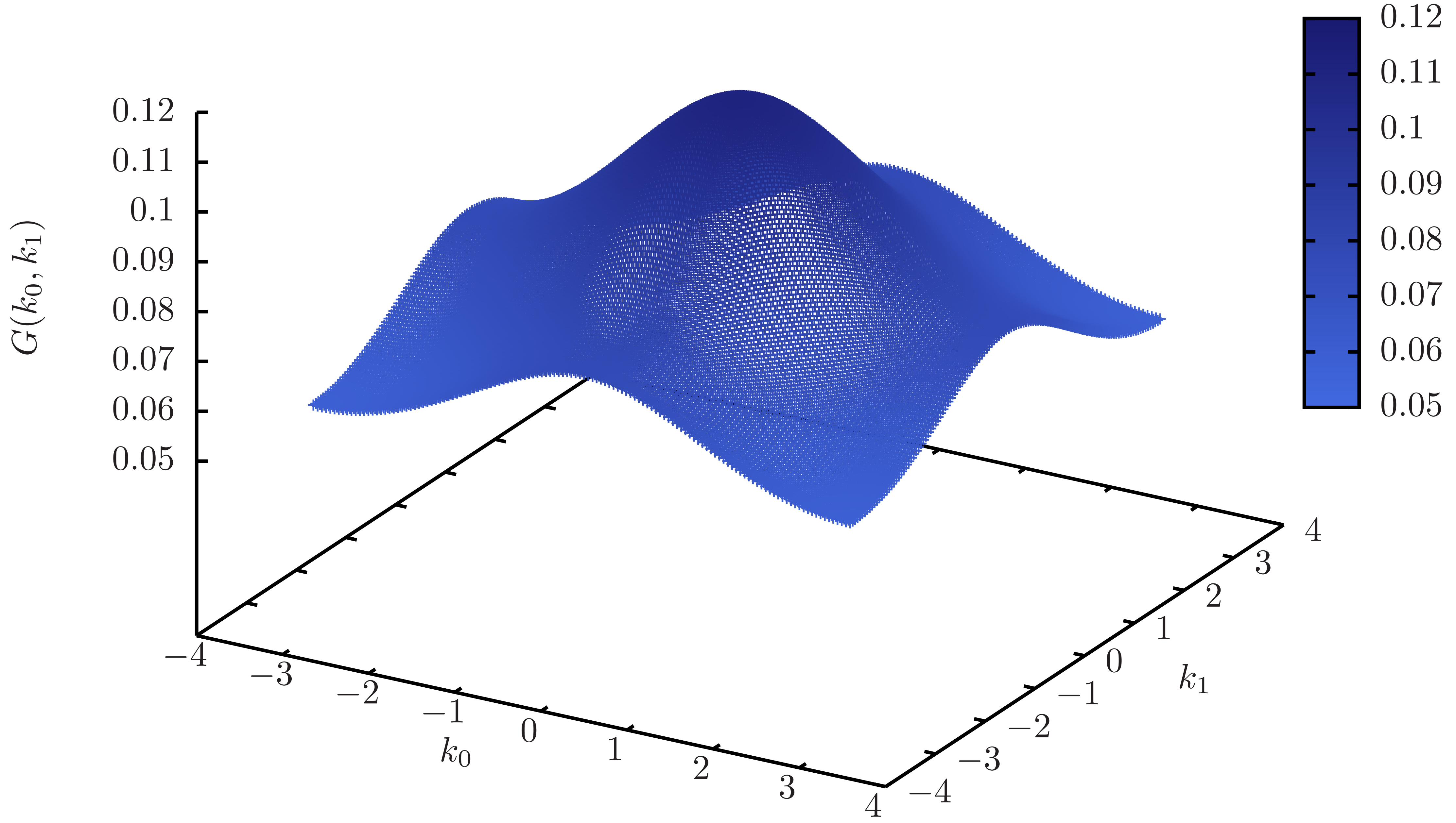


NUMERICAL SIMULATIONS: RESULTS

PROPAGATOR IN EUCLIDEAN SPACE:

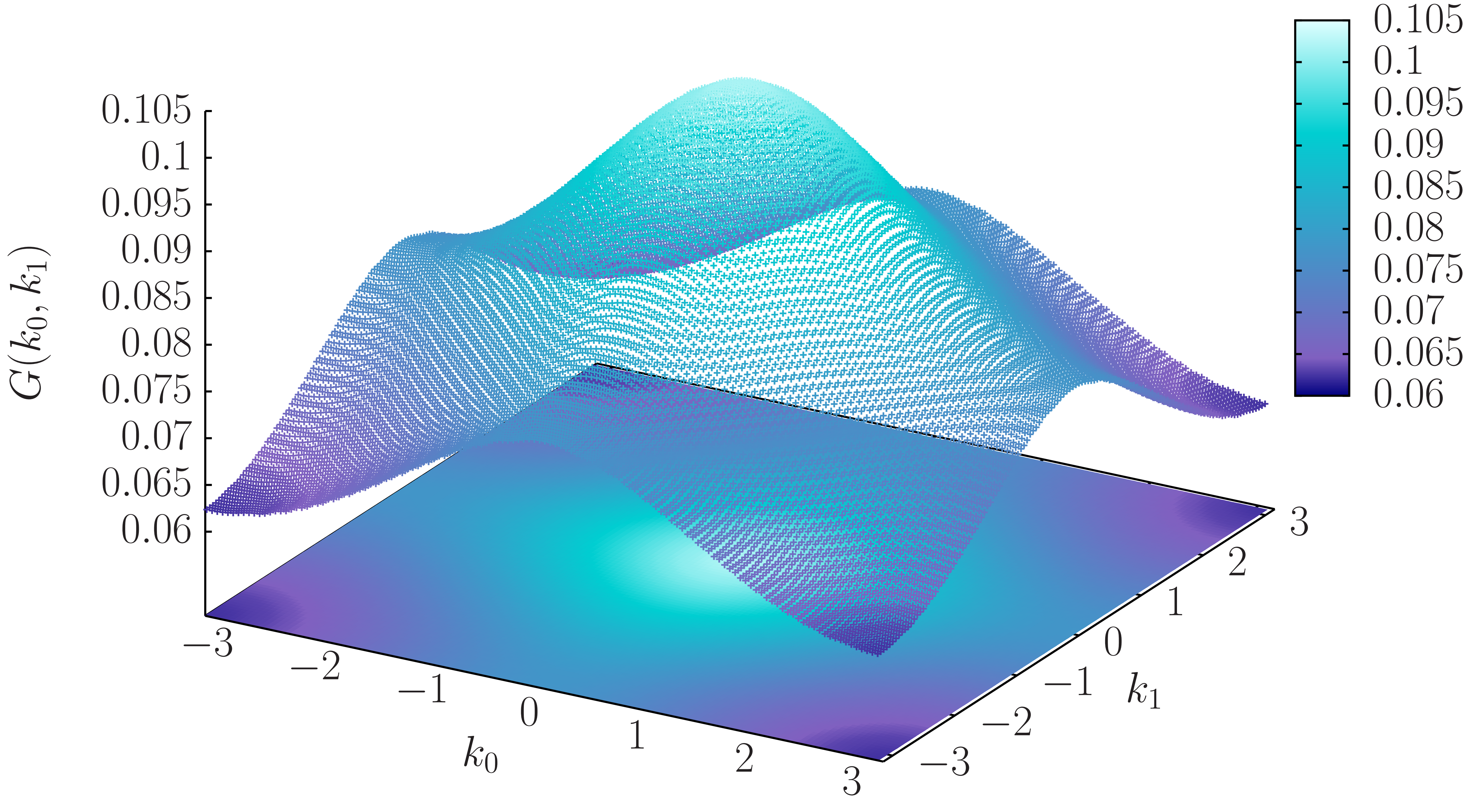
consistent with Stochastic Quantisation

Theoretical value



$$G(k, k_0) = \frac{1}{\frac{4}{a^2}\sin^2\left(\frac{ak_0}{2}\right) + \frac{4}{a^2}\sin^2\left(\frac{ak_1}{2}\right) + m^2}$$

Computed value

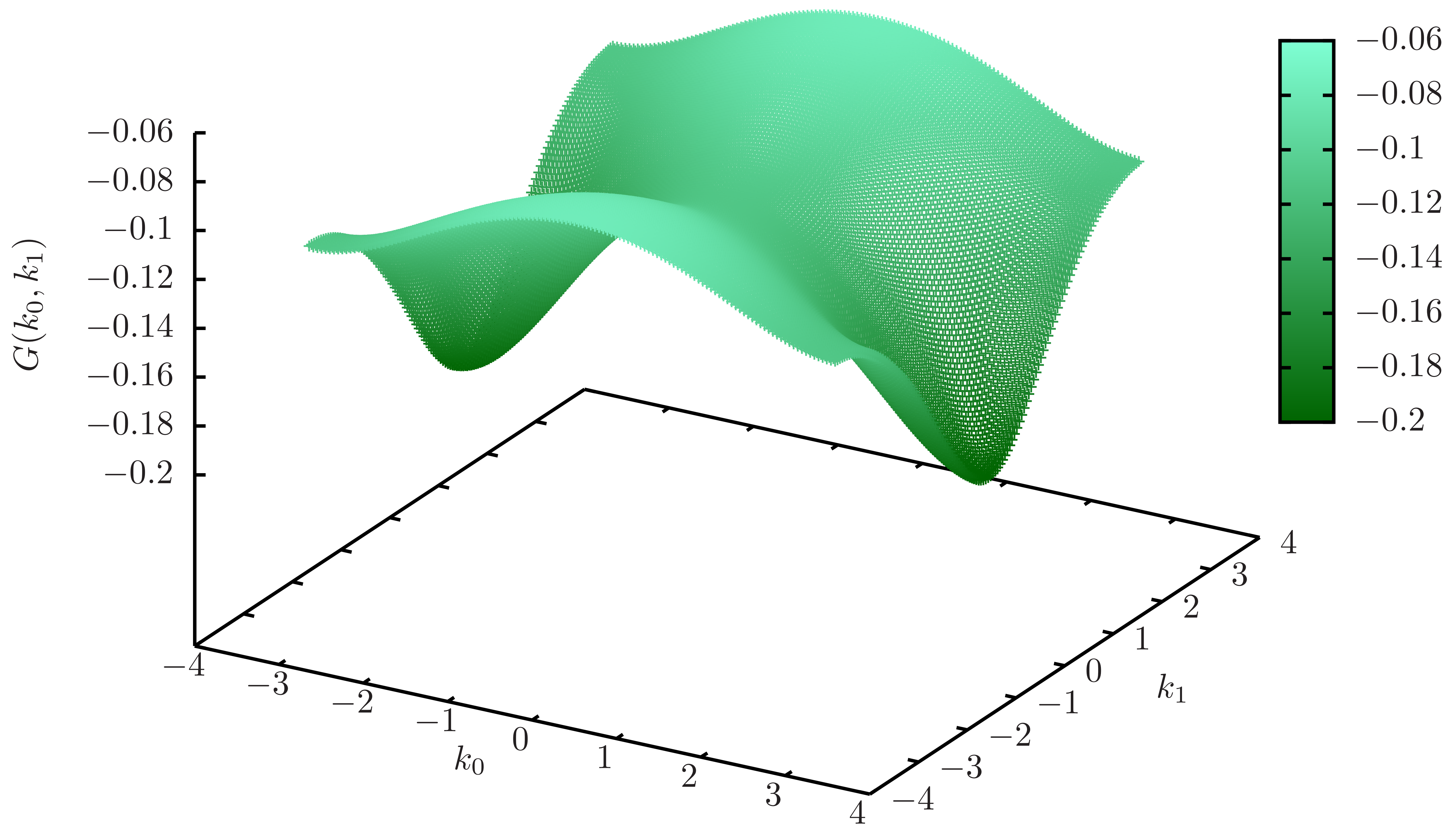


m=3, a=1.0, λ=0.01, La=128

NUMERICAL SIMULATIONS: RESULTS

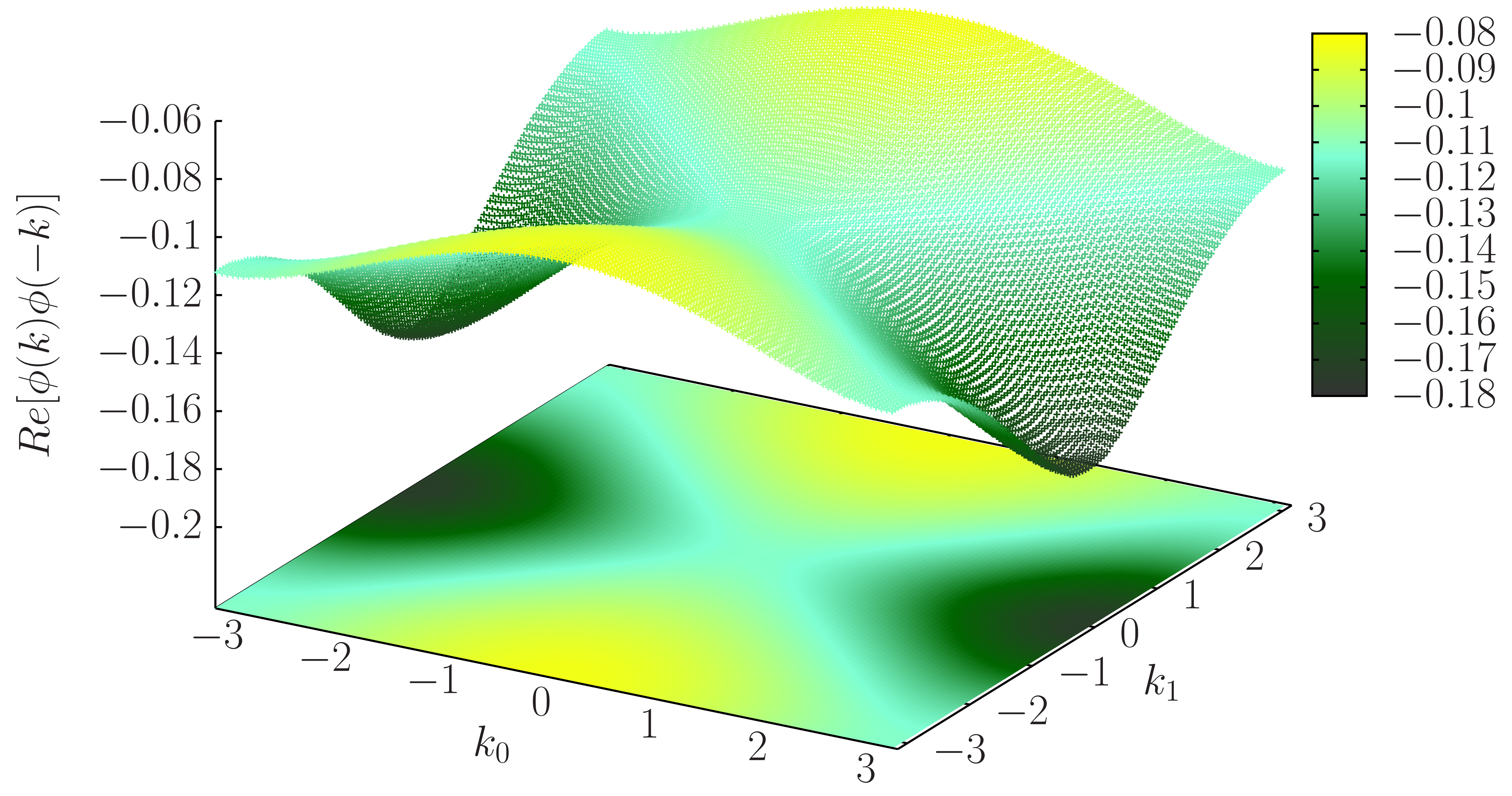
PROPAGATOR IN MINKOWSKI SPACE

Theoretical value



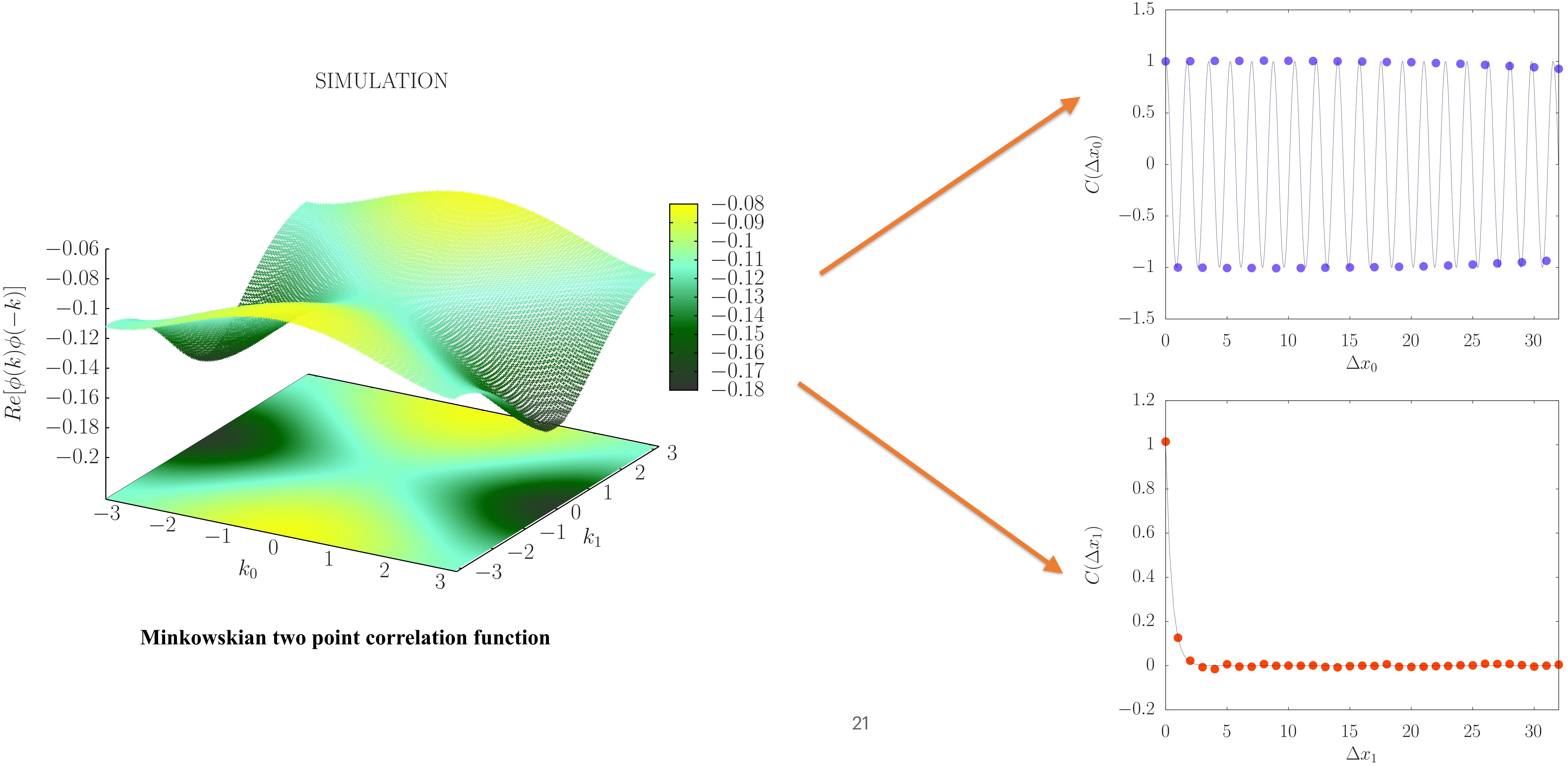
$$G(k_0, k_1) = \frac{1}{\frac{4}{a^2}\sin^2\left(\frac{ak_0}{2}\right) - \frac{4}{a^2}\sin^2\left(\frac{ak_1}{2}\right) - m^2}$$

Computed value



m=3, a=1.0, λ=0.01, La=128

PROPAGATOR IN MINKOWSKI SPACE



Minkowskian two point correlation function

5 CONCLUSIONS AND PERSPECTIVES

We have found a new dynamics for relativistic quantum field fluctuations that is able to produce sensible results.

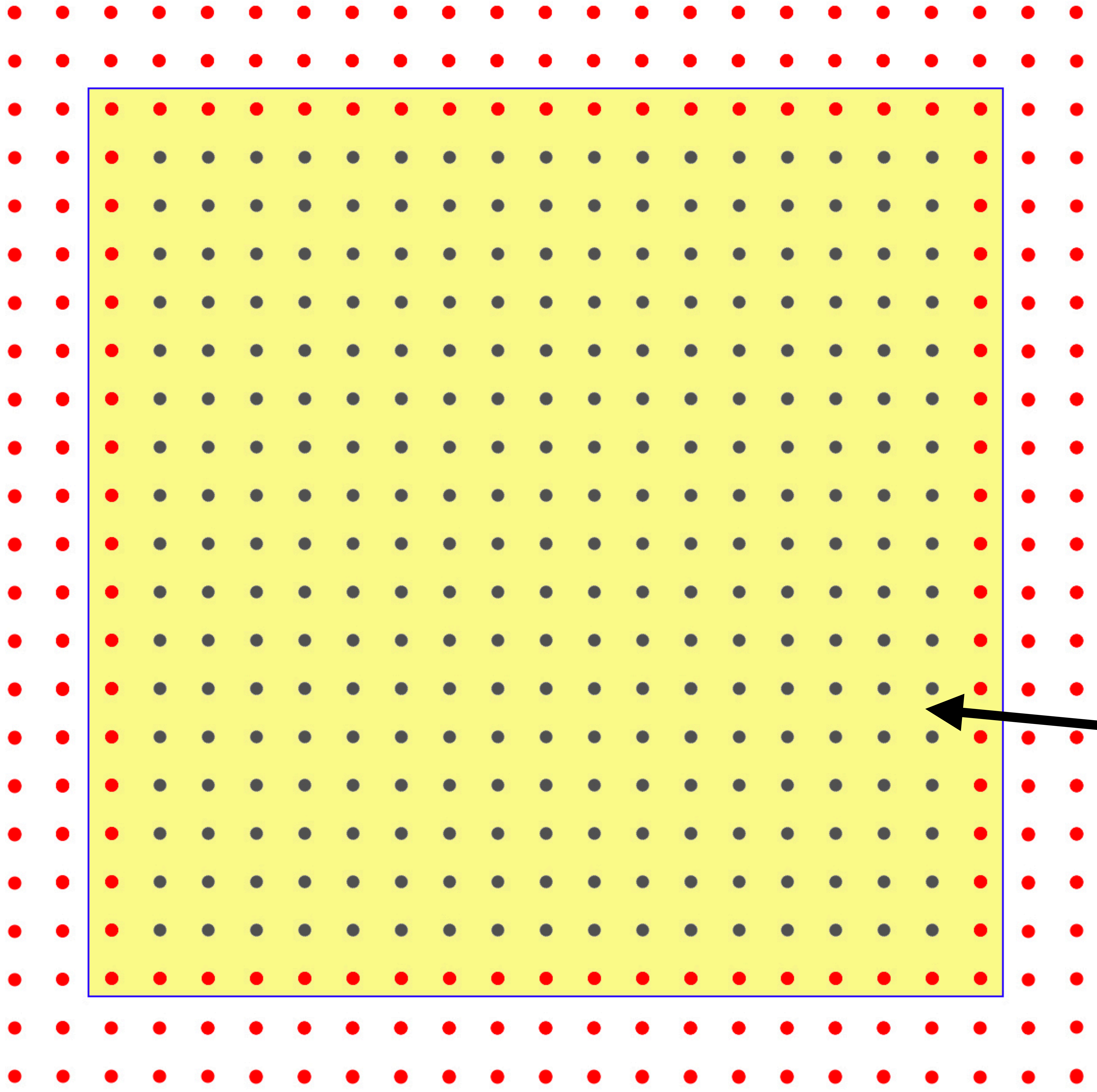
- Deterministic dynamic to describe quantum fluctuations
- New additional variables: conjugate momenta
- New time parameter: different from observers time
- Global constraint: generalised action is conserved
- Preserves the causal structure of space-time

Thank you!

4 **NUMERICAL SIMULATIONS**

$\lambda\phi^4$ scalar field theory in 1+1 dimensions

Boundary conditions: FRINGE BOUNDARIES CONDITION



WE CANNOT USE PERIODIC BOUNDARIES BECAUSE THEY DON'T PRESERVE CAUSALITY

We have to introduce some ad hoc boundary conditions:
ABSORBING BOUNDARIES

$$\ddot{\phi}(x_{\mathbf{i}},\tau) = -\Delta_0\phi + \Delta_1\phi - m^2\phi(x_{\mathbf{i}},\tau) - \lambda\phi^3(x_{\mathbf{i}},\tau)$$

$$\ddot{\phi}(x_{\mathbf{i}},\tau) = f_{\text{damping}}\left(-\Delta_0\phi + \Delta_1\phi - m^2\phi(x_{\mathbf{i}},\tau)\right) - \lambda\phi^3(x_{\mathbf{i}},\tau)$$