Numerical simulation of fractional topological charge in SU(N) gauge theory coupled with $\mathbb{Z}_N$ 2-form gauge fields

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lattice conference 2024 @ Liverpool U.

2024/08/01

- work in progress
  MA, O. Morikawa (RIKEN)

- Topology of SU(N) lattice gauge theories coupled with $\mathbb{Z}_N$ 2-form gauge fields
  MA, O. Morikawa, S. Onoda, H. Suzuki and Y. Tanizaki
  JHEP **2023**, 118 (2023) arXiv:2303.10977[hep-th]

Symmetry and Anomaly Ⅰ

- Classical Theory：Symmetry ⟷ Conservation low （Noether Theorem）
- Quantum Theory：The conservation low may be broken (Anomaly).
  - ➢ Focus on the Partition function,

$$Z[A] = \int [\mathcal{D}(\text{field})] \; e^{S[\text{field}, A]}.$$

  - ➢ How to distinguish the anomaly : Whether the Z is invariant or not under a transformation

$$Z' \stackrel{?}{=} Z$$

$$\rightarrow Z'[A + \partial\theta] = \int [\mathcal{D}(\text{field})] \; e^{S[\text{field}, A+\partial\theta]}$$

$$= \underbrace{e^{\mathcal{A}[\theta, A]}}_{\text{'t Hooft anomaly}} \; \underbrace{\int [\mathcal{D}(\text{field})] \; e^{S[\text{field}, A]}}_{= Z}.$$

Symmetry and Anomaly Ⅱ

- ➢ We can predict the low energy dynamics of the gauge theory.

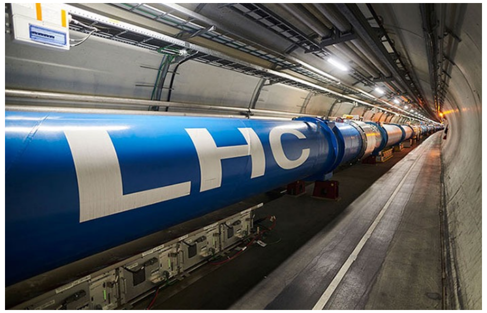
  ※ Gauge theory : A theory which describes the Standard Model of particles

- ✓ e. g., we decided the theory for the strong interaction is the SU(3) gauge theory because the theory and the experiment are well matched.

Particle Theory

Predict

Particle Experiment



https://www.icepp.s.u-tokyo.ac.jp/information/20220426.html

High — Low

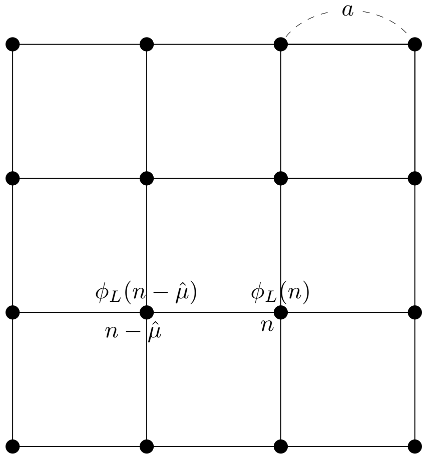
Energy

Recent Developments in Anomalies

- Recently, Gaiotto et al. has extended the concept of symmetry. : Generalized Symmetry
  (Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148[hep-th])
  - ➢ By anomalies with generalized (and discrete) symmetries, the low energy dynamics of gauge theories has been discussed. (Gaiotto, Kapustin, Komargodski, Seiberg, arXiv:1703.00501[hep-th])
  - ➢ Many new anomalies have been discovered and related studies has been done.
    - ✓ Yamaguchi, arXiv:1811.09390[hep-th]
    - ✓ Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389[hep-th]
    - ✓ Honda, Tanizaki, arXiv:2009.10183[hep-th]
    - ✓ etc.

☆Our Motivation : Understand these anomalies in the lattice field theory where we treat the regularization well.

Lattice Gauge Theory



Anomaly of the $SU(N)$ gauge theory with θ term

- The $SU(N)$ gauge theory with the θ term has the time reversal $(\mathcal{T})$ symmetry at $\theta = \pi$.

$$Z = \int \mathcal{D}a\ e^{S[a]} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\theta Q[a]}, \quad Q \in \mathbb{Z}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z' = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(-Q[a])} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(+Q[a])} \underbrace{e^{-i2\pi Q[a]}}_{=1} = Z$$

- Then, we construct the $SU(N)$ gauge theory with the higher form symmetry ($\mathbb{Z}_N$ 1-form symmetry). This means we couple $\mathbb{Z}_N$ 2-form gauge field to the theory.
  - ➢ The topological charge (TC) becomes fractional, so it is not invariant under the $\mathcal{T}$ transformation.

Important!!

$$e^{-i2\pi Q} \neq 1$$

  - ➢ This theory at $\theta = \pi$ has the mixed anomaly between the $\mathbb{Z}_N$ 1-form and $\mathcal{T}$ symmetry.

$\mathbb{Z}_N$ 1-form Global Transformation on the Lattice

- Lattice $SU(N)$ gauge theory, the action is

$$S_W[U_l] \equiv \sum_p \beta\left[\mathrm{tr}\left(\mathbb{1} - U_p\right)\right].$$

➢ Center transformation ($\mathbb{Z}_N$ 1-form global transformation) on the lattice acts on the link variables,

$$U_l \mapsto e^{\frac{2\pi i}{N}k} U_l, \quad W(C) \mapsto e^{\frac{2\pi i}{N}k} W(C).$$

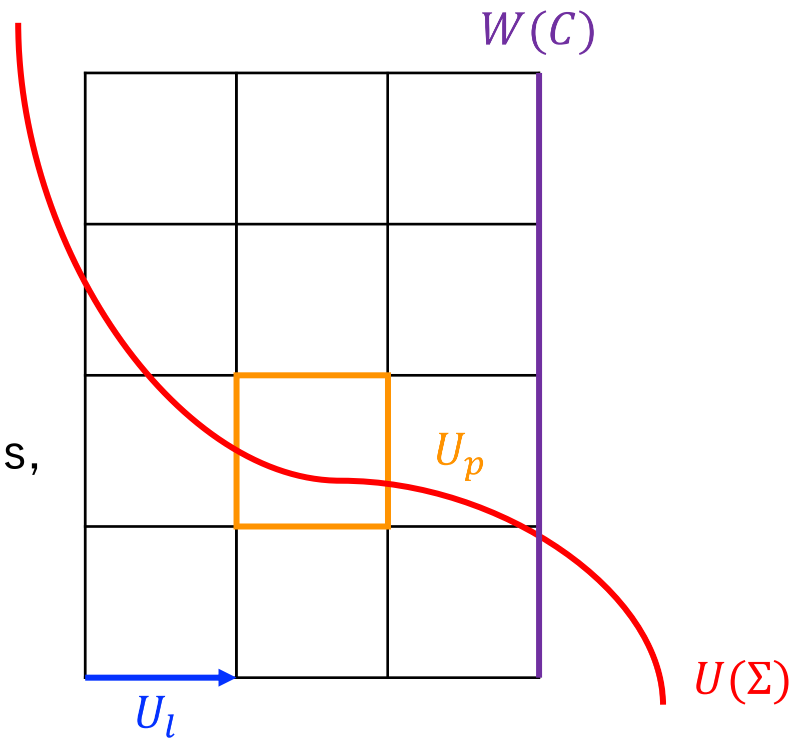
※ Under this 1-form global transformation the Wilson line changes,

$$\langle U_\alpha(\Sigma) W(C) \cdots \rangle = e^{-i\alpha} \langle W(C) \cdots \rangle$$

➢ The transition function satisfies the cocycle condition still.

$$v_{n-\hat{\mu},\nu}(x) v_{n,\mu}(x) v_{n,\nu}(x)^{-1} v_{n-\hat{\nu},\mu}(x)^{-1} = \mathbb{1}.$$

transition function



$\mathbb{Z}_N$ 1-form Gauge Transformation on the Lattice

- Gauging the center symmetry, the action becomes

$$S_{\mathrm{W}}[U_\ell, B_p] = \sum_p \beta \left[\mathrm{tr}\left(\mathbb{1} - e^{-\frac{2\pi i}{N} B_p} U_p\right)\right].$$

➢ Invariant under the $\mathbb{Z}_N$ 1-form gauge transformation,

$$U_l \mapsto e^{\frac{2\pi i}{N}\lambda_l} U_l, \quad B_p \mapsto B_p + (\mathrm{d}\lambda)_p.$$

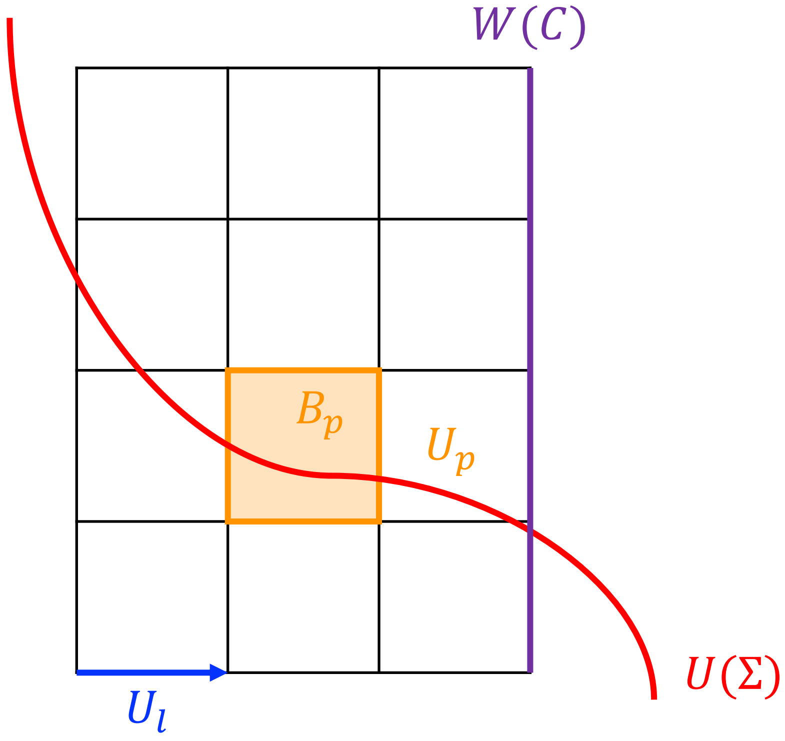
➢ The cocycle condition is relaxed,

$$\tilde{v}_{n-\hat{\nu},\mu}(n)\tilde{v}_{n,\nu}(n)\tilde{v}_{n,\mu}(n)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(n)^{-1} = e^{\frac{2\pi i}{N} B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}\mathbb{1}.$$

※ ‘t Hooft twisted boundary condition

$$U(n + L\hat{\nu}, \mu) = g^{-1}_{n,\mu} U(n, \mu) g_{n+\hat{\mu},\nu}$$

$$g^{-1}_{n+L\hat{\nu},\mu} g^{-1}_{n,\nu} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N} z_{\mu\nu}}, \quad z_{\mu\nu} = \sum_p B_p \bmod N.$$



Topological Charge on the Lattice

- How to calculate the topological charge Q,

$$Q = -\frac{1}{24\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{f(n,\mu)} \mathrm{d}^3x \,\mathrm{tr}\left[(v_{n,\mu}^{-1}\partial_\nu v_{n,\mu})(v_{n,\mu}^{-1}\partial_\rho v_{n,\mu})(v_{n,\mu}^{-1}\partial_\sigma v_{n,\mu})\right]$$

$$-\frac{1}{8\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{p(n,\mu,\nu)} \mathrm{d}^2x \,\mathrm{tr}\left[(v_{n,\mu}\partial_\rho v_{n,\mu}^{-1})(v_{n-\hat{\mu},\nu}^{-1}\partial_\sigma v_{n-\hat{\mu},\nu})\right].$$

  - ➢ $v_{n,\mu}(x)$ is the gauge translation function (transition function).
- On the lattice, topological values are ill-defined.
  - ➢ Restricting the size of plaquette (admissibility condition), Lüscher constructed integral TC on the lattice (Lüscher, Commun. Math. Phys. 85 (1982)).
  - ➢ We aim to construct the fractional TC on the $SU(N)$ lattice by extended the Lüscher's topological charge.
    - ✓ Itou, arXiv:1811.05708[hep-th]
    - ✓ Anosova, Gattringer, Göschl, Sulejmanpasic, Törek, arXiv:1912.11685 [hep-lat]

Transition Function for Fractional TC

- Coupling $\mathbb{Z}_N$ 2-form field to the theory, the structure of fiber bundle becomes rich.

$$\tilde{v}_{n-\hat{\nu},\mu}(n)\tilde{v}_{n,\nu}(n)\tilde{v}_{n,\mu}(n)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(n)^{-1} = e^{\frac{2\pi i}{N}B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}\mathbb{1}.$$

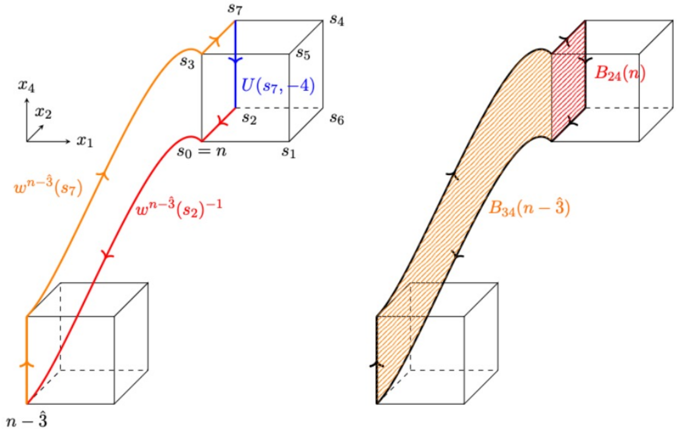
- We find that the $\mathbb{Z}_N$ 1-form gauge invariance plays the center role.

Admissibility condition

$$\|\mathbb{1} - \tilde{U}_p(n)\| < \varepsilon,$$

$$\tilde{U}_p \equiv e^{-\frac{2\pi i}{N}B_{\mu\nu}(n)}U_p.$$

Components of transition function



Construction of $u^{n-\hat{3}}_{s_7 s_2}$ Construction of $\tilde{u}^{n-\hat{3}}_{s_7 s_2}$

Fractional Topological charge

- By the $\mathbb{Z}_N$ 1-form invariant transition function, we calculate TC,

$$z_{\mu\nu} = \sum_{p\in(T^2)_{\mu\nu}} B_p \qquad \bmod N,$$

$$Q_{\mathrm{top}} = -\frac{1}{N}\int_{T^4}\frac{1}{2}P_2(B_p) \bmod 1 \in -\frac{1}{N}\frac{\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}}{8} + \mathbb{Z},$$

$$P_2(B_p) = B_p \cup B_p + B_p \cup_1 \mathrm{d}B_p\,.$$

- In the $U(1)$ lattice gauge theory, we make sure that (cf. Abe, Morikawa, Suzuki, arXiv:2210.12967[hep-th])

$$Q_{\mathrm{top}} = \frac{1}{32\pi^2}\sum_{n\in\Lambda}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\tilde{F}_{\mu\nu}(n)\tilde{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) \in \frac{1}{N^2}\mathbb{Z} + \mathbb{Z}.$$

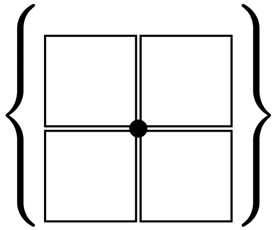
Numerical Simulation of Topological Charge

- Simulate the topological charge on the lattice
  - ☐ Teper, *Nucl.Phys.B Proc.Suppl.* 83 (2000)
  - ☐ Ali Khan, *et.al, Phys. RevD.* 64.114501 (2001)
  - ☐ Itou, *JHEP,* 05, 09 3(2019)
  - ☐ Mon 15:15 (Antonio Gonzalez-Arroyo), Mon 15:35 (Ivan Soler), Tue 14:05 (Benjamin Jaeger)

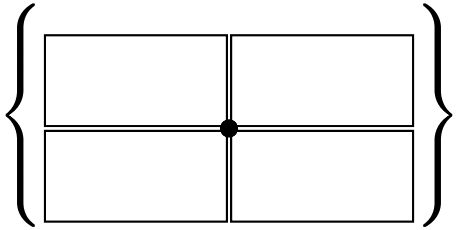
← To get the “integer” value

- ✓ Improved topological operator

$$Q_{\rm imp} = \sum_{x_n} \left\{ c_0 Q_L^P(x_n) + c_1 Q_L^R(x_n) \right\}$$

$C^P_{\mu\nu} = \mathrm{Im}$ 

- ✓ Cooling step
  - ☐ Weisz*, Nucl. Phys.* B212,1 (1983)
  - ☐ Lüscher, *Commun. Math. Phys*. 97, 59(1985)

$C^P_{\mu\nu} = \mathrm{Im}$ 

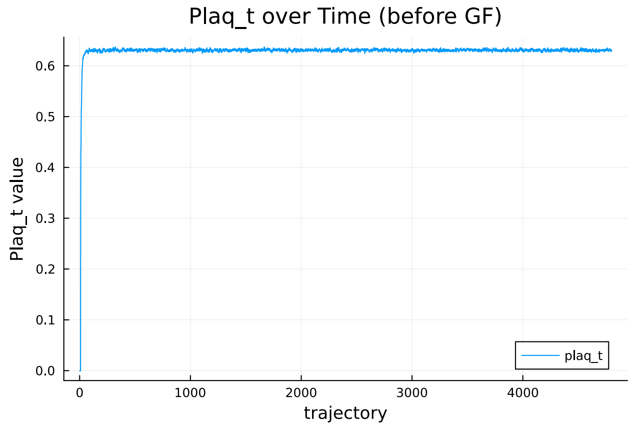
Integer topological charge

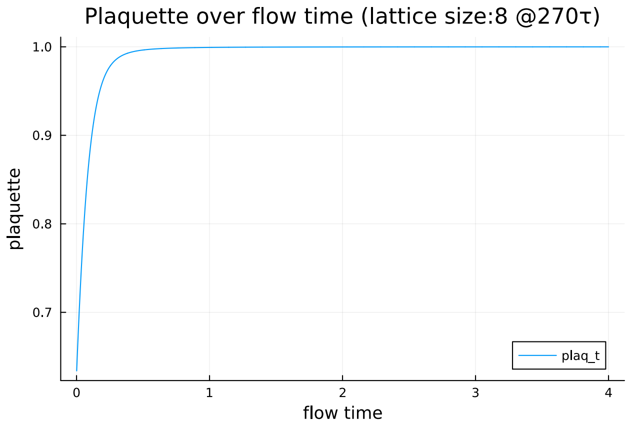
✓ Using LatticeQCD.jl (https://github.com/akio-tomiya/LatticeQCD.jl)

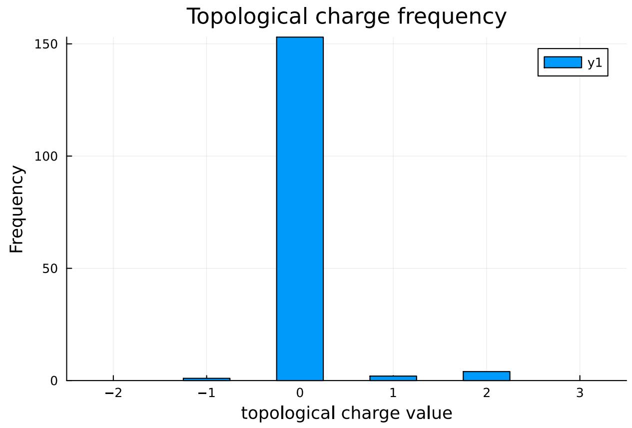
① Get gauge configurations by Hybrid Monte Carlo method ($\beta = 6.0$, lattice size 8)
(Schaefer, Sommer, Virotta, arXiv:1009.5228 [hep-lat])

② Take smooth configurations by gradient flow

③ Calculate topological charge at each trajectory

①


②


③


Fractional topological charge

Couple $\mathbb{Z}_N$ 2-form gauge fields to the action

$$S_W[U_l] \equiv \sum_p \beta \left[\mathrm{tr}\left(\mathbb{1} - U_p\right) + \mathrm{c.c.}\right]$$

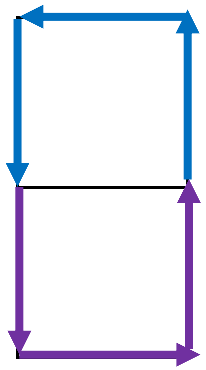
$$S_\mathrm{W}[U_\ell, B_p] = \sum_p \beta \left[\mathrm{tr}\left(\mathbb{1} - e^{-\frac{2\pi i}{N} B_p} U_p\right) + \mathrm{c.c.}\right],$$

$$\sum_{p \in \partial f} B_p = 0 \bmod N$$

➢ By this action, do ①〜③ step

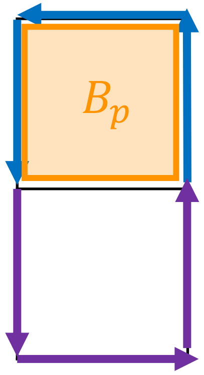
※ Step ①, ② includes staples,
so we also couple $\mathbb{Z}_N$ 2-form gauge fields with staples.

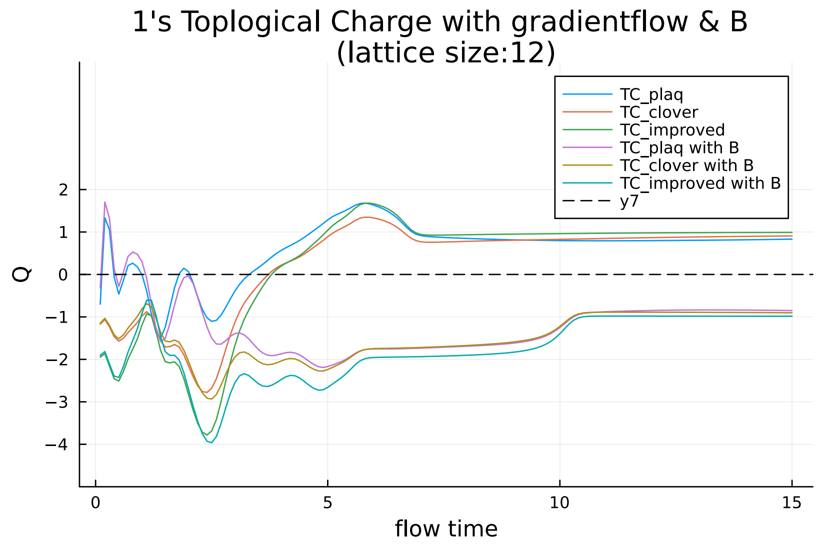
$$V_\mu(n) = \sum_{\nu=1,\nu\neq\mu}^{4}$$



⇒

$$V'_\mu(n) = \sum_{\nu=1,\nu\neq\mu}^{4}$$





Conclusion & Future Work

☆ Conclusion

- Construct the fractional topological charge on the lattice
- Simulate the fractional topological charge

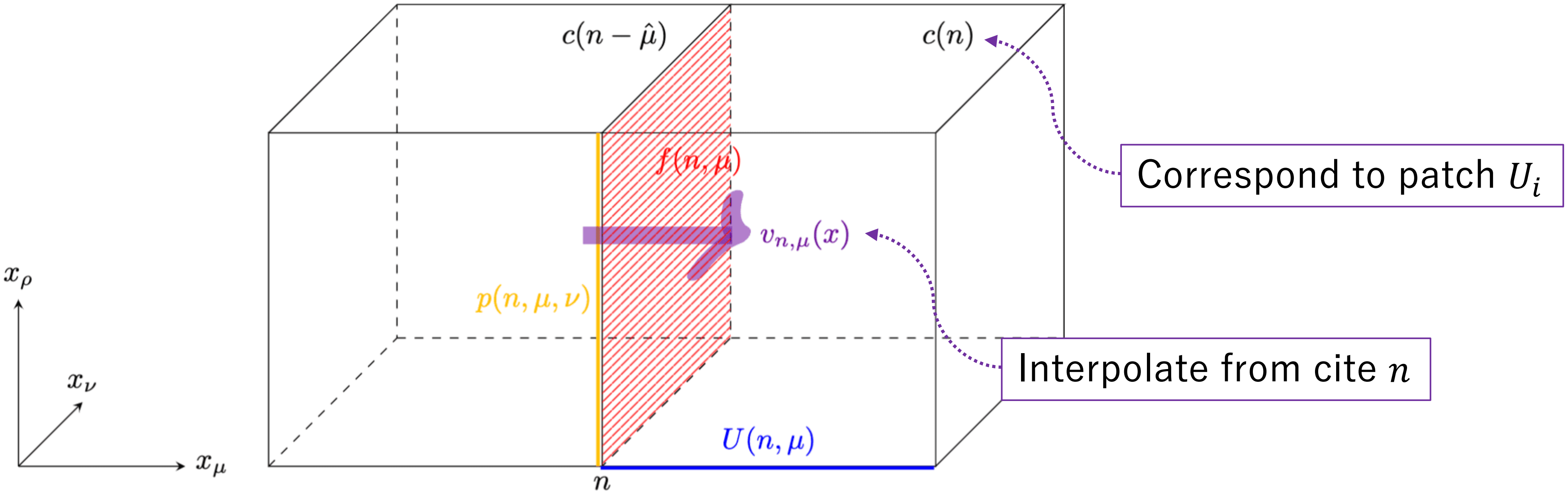
☆Future Work

- Application to Berry phase
  (Fukui, *et.al, JPSJ*.74.1674(2005))
- Phase structure vs Sign problem
  - Gaiotto, *et.al*, *JHEP,* 05, 091 (2017)
- Numerical simulation to other lattice
  - Phillips, Stone, *Commun. Math. Phys.* 103, 599, 636 (1986)
  - Budich, Ardonne, Phys. Rev. B **88**, 035139 (2013)

Back Up

To Construct Topological Charge on the Lattice

- The manifold is divided by hyper cubes $c(n)$.
- e. g., in the 3d,



Anomaly I

- Again, the action on the lattice is

$$S[U_l, B_p] \equiv -S_W[U_l, B_p] + i\theta Q_{\rm top}[U_l, B_p].$$

- The topological charge is

$$Q_{\rm top} = -\frac{1}{N}\int_{T^4}\frac{1}{2}P_2(B_p) + \mathbb{Z} \equiv {\rm frac}[B_p] + {\rm int}[U_l, B_p].$$

  ✧ Manifestly invariant under the $\mathbb{Z}_N$ one-form gauge transformation

➢ We discuss the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and θ shift.

Anomaly II

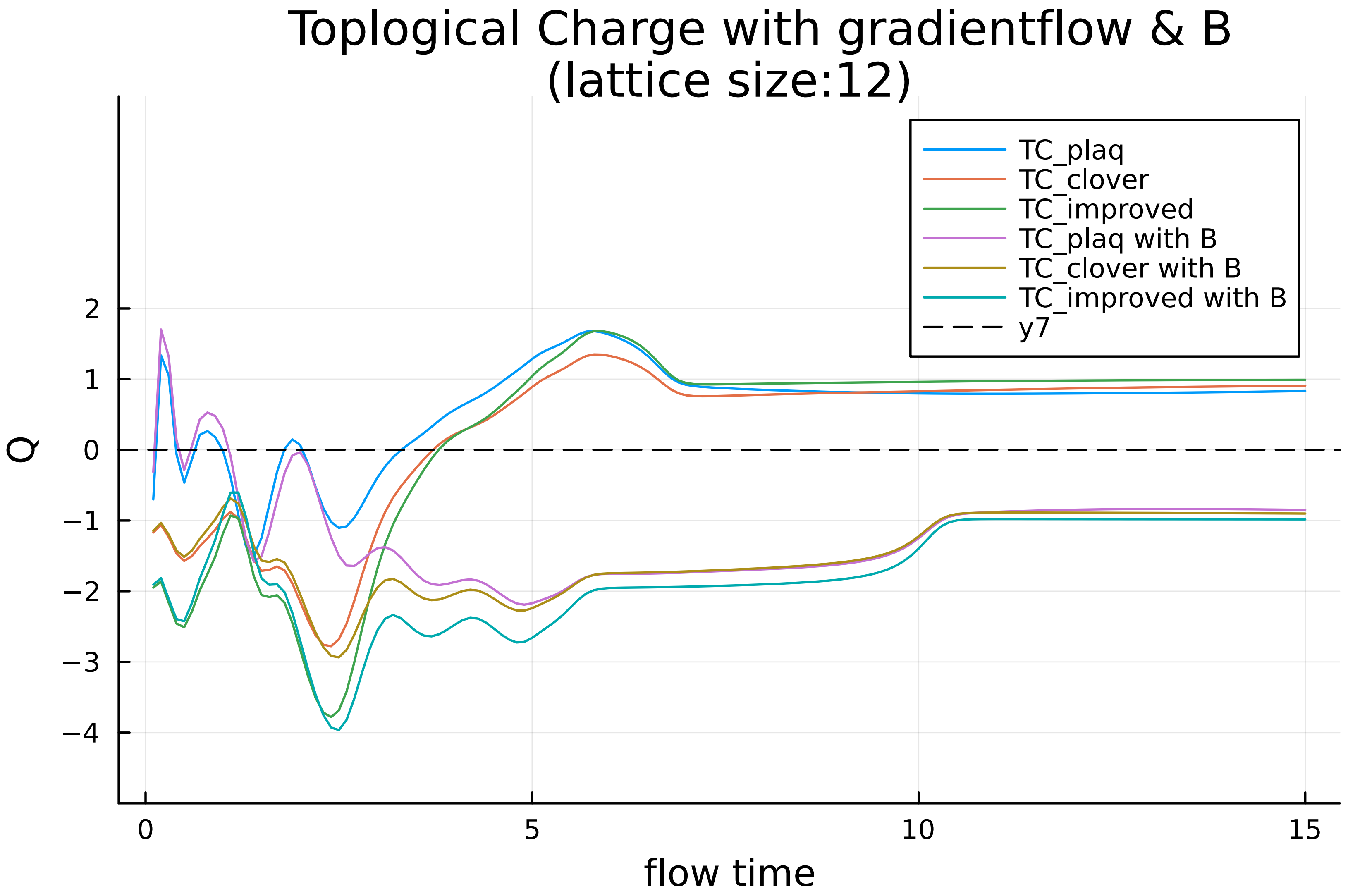
- At $\theta = \pi$, the partition function is, under $\mathcal{T}$ transformation,

$$Z[B_p] = \int \mathcal{D}U_l \; e^{S[U_l,B_p]} = \int \mathcal{D}U_l \; e^{-S_W[U_l,B_p]} e^{i\theta Q_{\mathrm{top}}[U_l,B_p]}$$

$$\xrightarrow[\theta=\pi,\ \theta \text{ shift.}]{} Z'[B_p] = \int \mathcal{D}U_l \; e^{-S_W[U_l,B_p]} e^{i\pi(-Q_{\mathrm{top}}[U_l,B_p])} = \int \mathcal{D}U_l \; e^{-S_W[U_l,B_p]} e^{i\pi Q_{\mathrm{top}}[U_l,B_p]} \underbrace{e^{-i2\pi Q_{\mathrm{top}}[U_l,B_p]}}_{=e^{-i2\pi \mathrm{int}[U_l,B_p]} e^{-i2\pi \mathrm{frac}[B_p]}}$$

$$= e^{-i2\pi \mathrm{frac}[B_p]} \underbrace{\int \mathcal{D}U_l \; e^{-S_W[U_l,B_p]} e^{i\pi Q_{\mathrm{top}}[U_l,B_p]}}_{=Z} \neq Z[B_p]$$

- This means that there is the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and θ shift.


Toplogical Charge with gradientflow & B
(lattice size:12)
TC_plaq
TC_clover
TC_improved
TC_plaq with B
TC_clover with B
TC_improved with B
y7
Q
2
1
0
−1
−2
−3
−4
0
5
10
15
flow time