The Affine Conjecture: Lattice Field Theory On Curved Manifolds

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Lattice 2024, Liverpool July 30

with G. Fleming (FNAL), N. Matsumoto (BU), E. Owen (BU), J. Y. Lin (Carnegie-Mellon)

Matching Curved Lattice to Anisotropic Tangent Planes (Poster: Flemming)

2D Ising Energy-momentum tensor in modular space (Talk:Matsumoto)

The Ising Model on Affine Plane (https://arxiv.org/abs/2209.1554: RB, Owen)

The Ising Model on $\mathbb{S}^2$ (https://arxiv.org/abs/2407.004590: R.B., E. Owen)

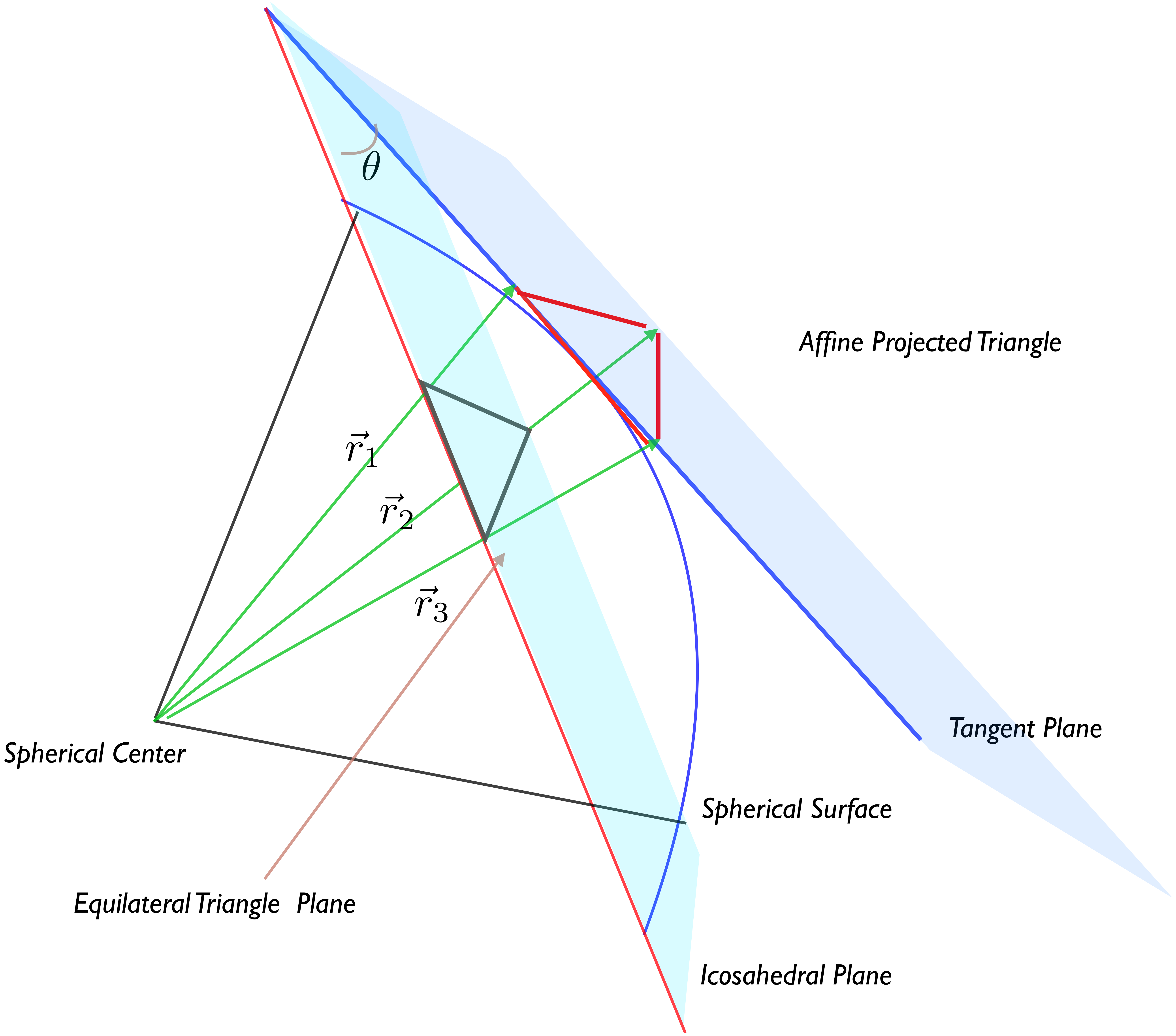
What is the Affine Conjecture?

- "The affine map between Regge's lattice geometry and the lattice couplings on each tangent provides an exact general solution to lattice field theory on smooth Euclidean manifolds in the continuum"

Motivation and background reading

1961 REGGE "General Relativity without Coordinates"
1974 WILSON "Confinement of Quarks" LATTICE QCD
1984 T D LEE et al " Lattice Gravity Near the Continuum"
1997 MALDACENA "Wyle transform to CFT at AdS Boundary"
....
2022-24 2d Ising Solution on the Affine Plane & Sphere

To O(a^2) the tangent plane is an Affine lattice on each tangent plane.



"The art of doing mathematics consists finding that special case which contains all the germs of generality."

David Hilbert Mathematician, Physicist, Philosopher*

*Author of Geometry and the Imagination

Classical Field Geometry

$$S = S_{EH} + S_M = \int dx \sqrt{g} [\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_M[\phi]]$$

$$\frac{\delta}{\delta g^{\mu\nu}(x)} \implies \quad R_{\mu\nu}(x) - \frac{1}{2} g_{\mu\nu}(x) R(x) = \kappa T_{\mu\nu}(x)$$

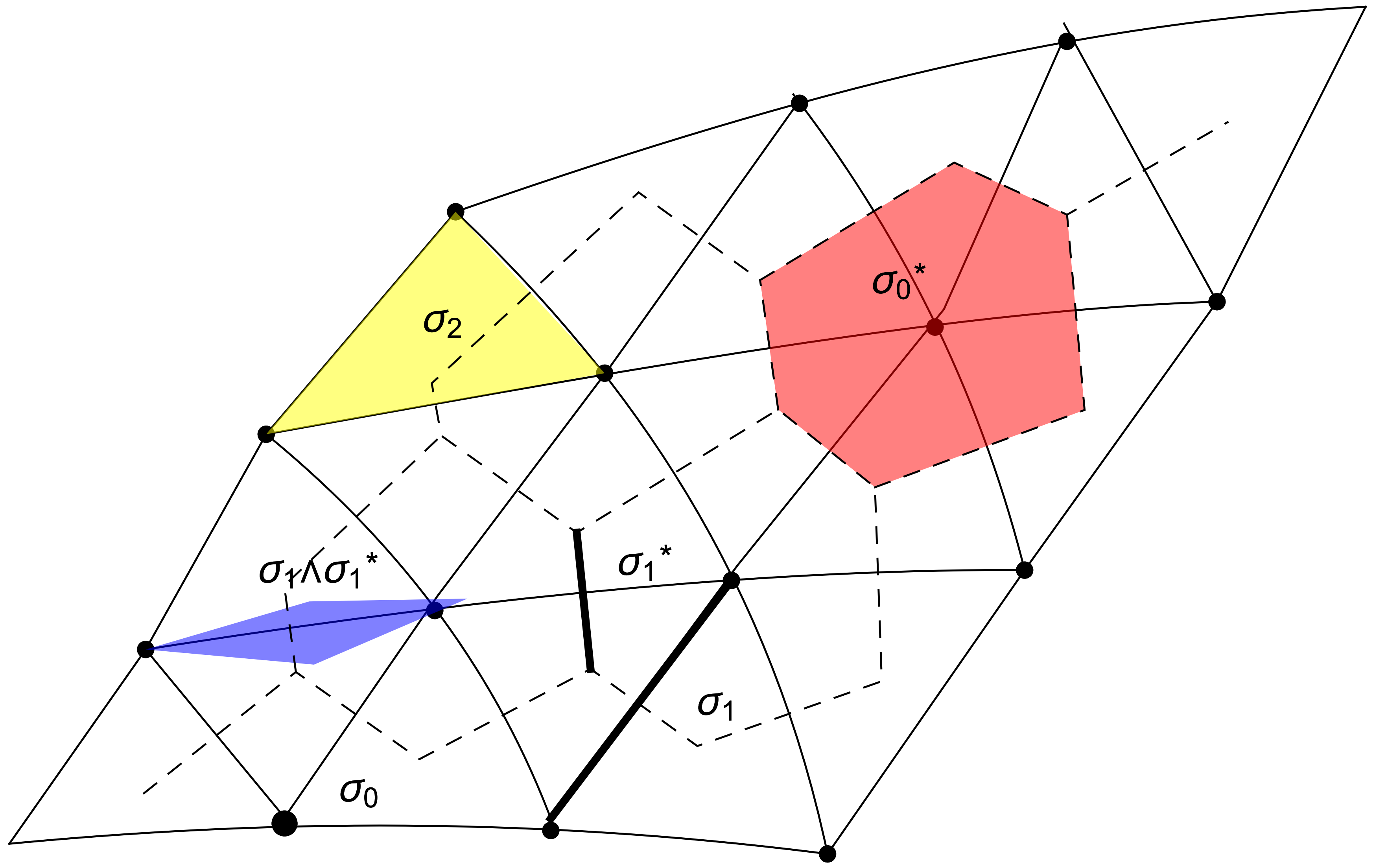
$$\frac{\delta \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n)\rangle}{\delta g^{\mu\nu}(x)} = \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n) T_{\mu\nu}(x)\rangle$$

How to put Quantum Fields on a Lattice?

Classical Gravity and Fields are piecewise elements on SAME simplicial graphs!

Classical Gravitation Metric Manifold

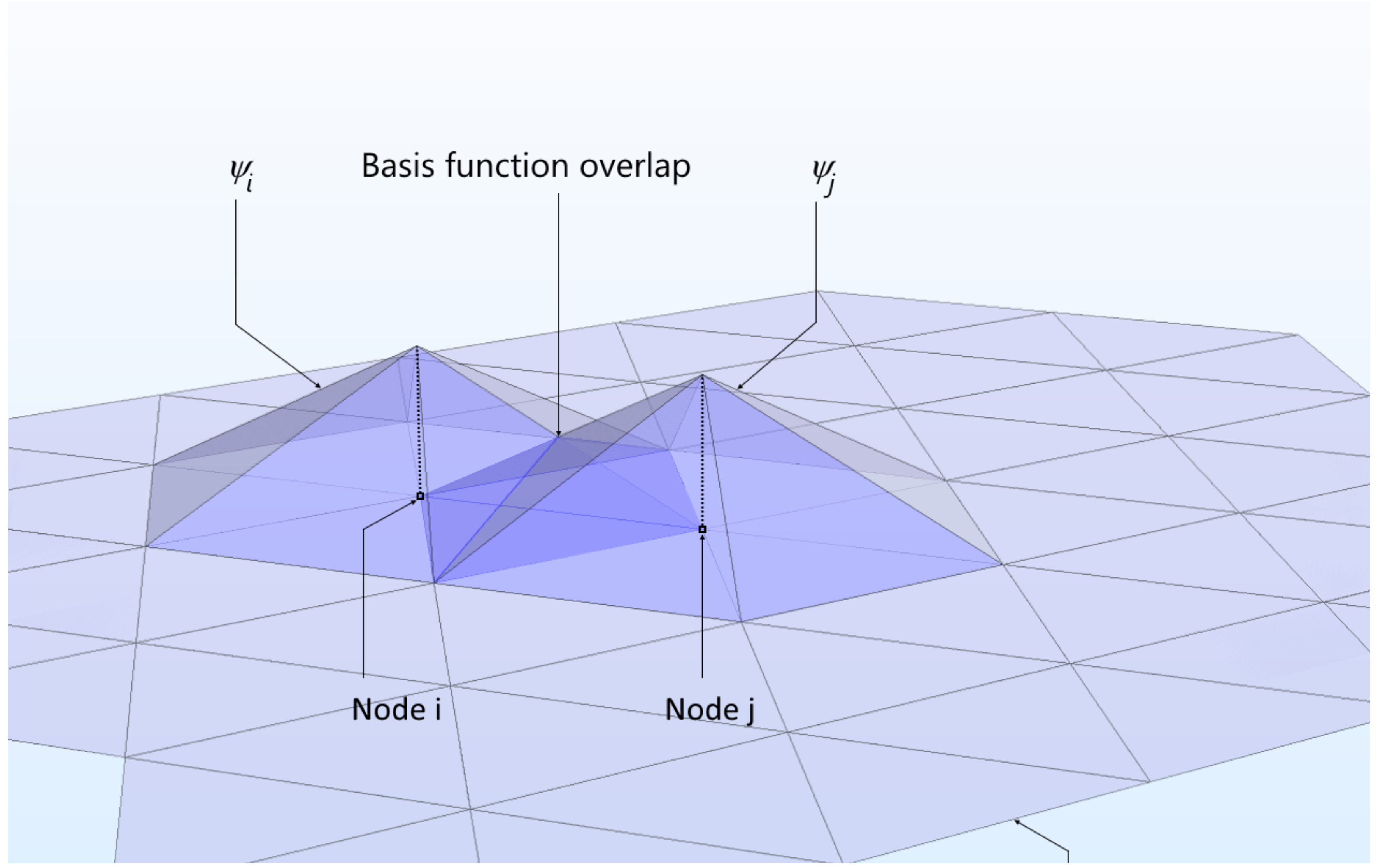
REGGE: Piecewise linear metric



$$(\mathcal{M}, g_{\mu\nu}(x)) \leftrightarrow (\mathcal{M}_\sigma, g_\sigma = \{l_{ij}\})$$

Classical Fields: PDEs

FEM: Piecewise linear fields



$$S = \frac{1}{2}[\sum_{\langle i,j\rangle} K_{i,j}(\phi_i - \phi_j)^2 + \lambda_0(\phi_x^2 - 1)^2]$$

Only Geometry as invariant Llength

$$ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$$

Only dimensionless fields and couplings plus FEM Discrete Exterior Calc map.

REGGE'S MANIFOLD :
"General Relativity without Coordinates" 1960

$$\{\mathcal{M}, g_{\mu\nu}\}$$

$$S_{EH} = \int d^d x \sqrt{g(x)} R(x)$$

$$\sqrt{g} R \sim A_h \epsilon_h \sqrt{g_\perp} \delta^2(x_\perp)$$

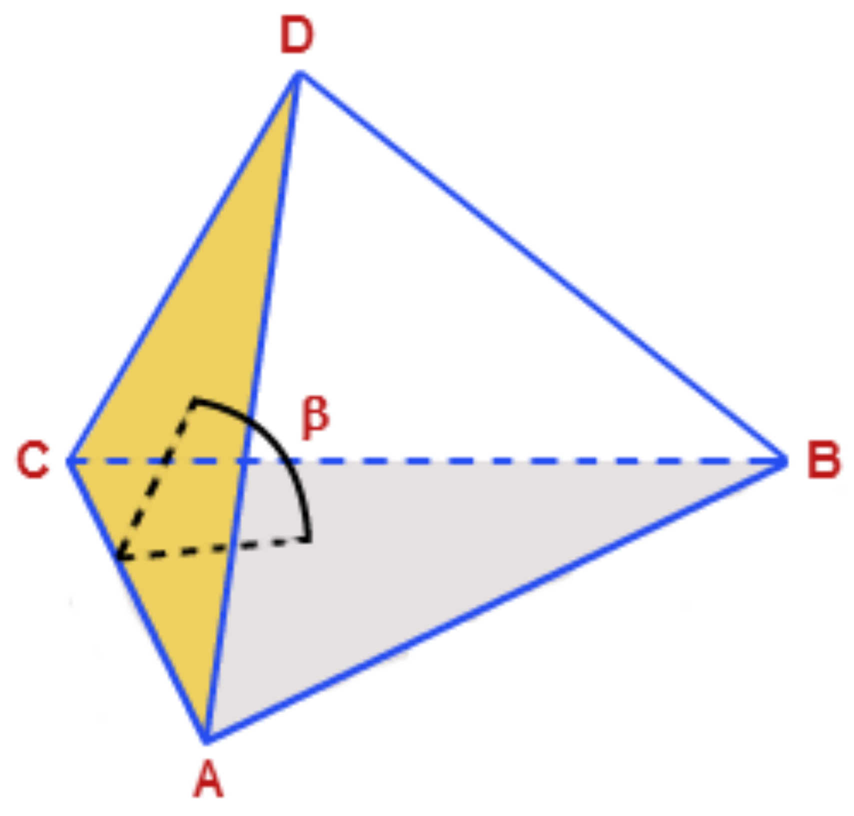
$$\{G, \ell_{ij}\}$$

dim = D -2

$$S_{Regge}[\ell_{ij}] = 2 \sum_{h \in G} A_h \epsilon_h$$

$$\epsilon_h = 2\pi - \sum_{h \in \sigma} \theta_{\sigma,h}$$

Dihedral Angle in simplex

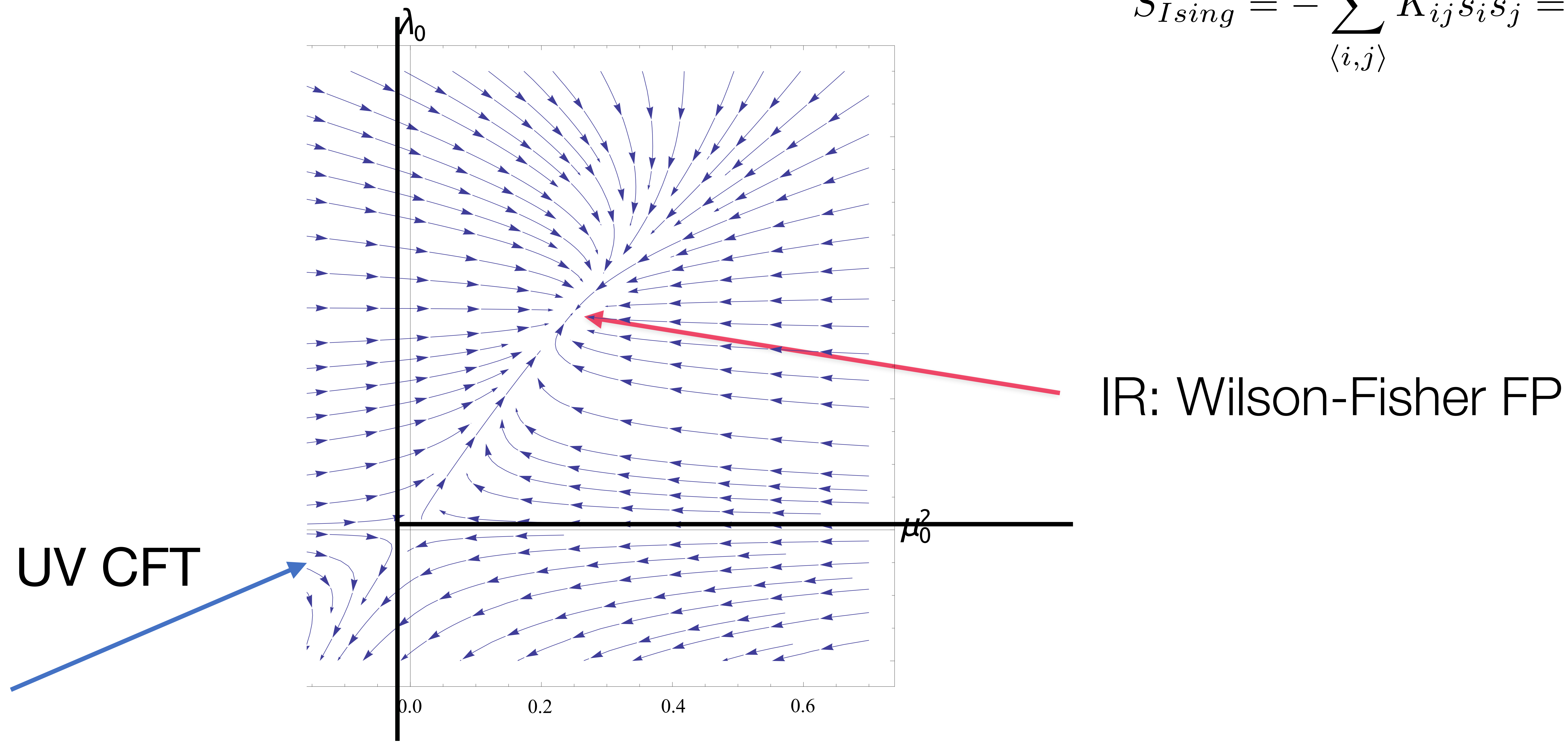


EOM

$$\frac{\partial S}{\partial \ell_{ij}} = \frac{\partial V_h}{\partial \ell_{ij}} \epsilon_h - \sum_{\sigma} \sum_{h \in \sigma} V_h \frac{\partial \theta_{\sigma,h}}{\partial \ell_{ij}}$$

= 0 by Shalafli ID

Scalar Phi4/Ising Model



(fixed $m^2 = -\mu_0^2/a^2, \lambda = \lambda_0/a^2$)

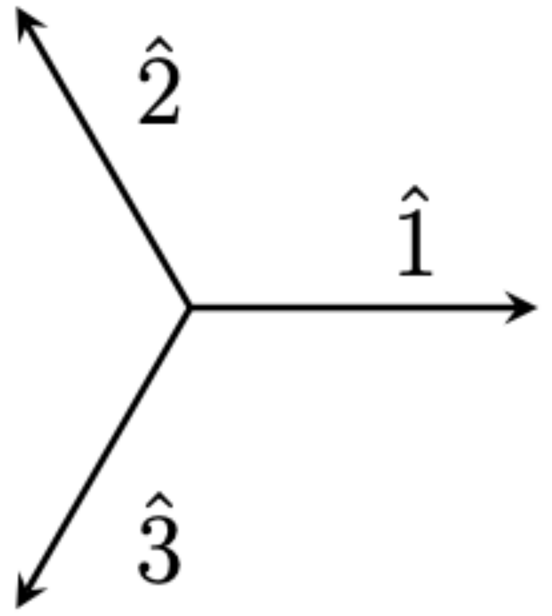
$$S_{Ising} = -\sum_{\langle i,j \rangle} K_{ij} s_i s_j = \frac{1}{2} \sum_{\langle i,j \rangle} K_{ij}(s_i - s_j)$$

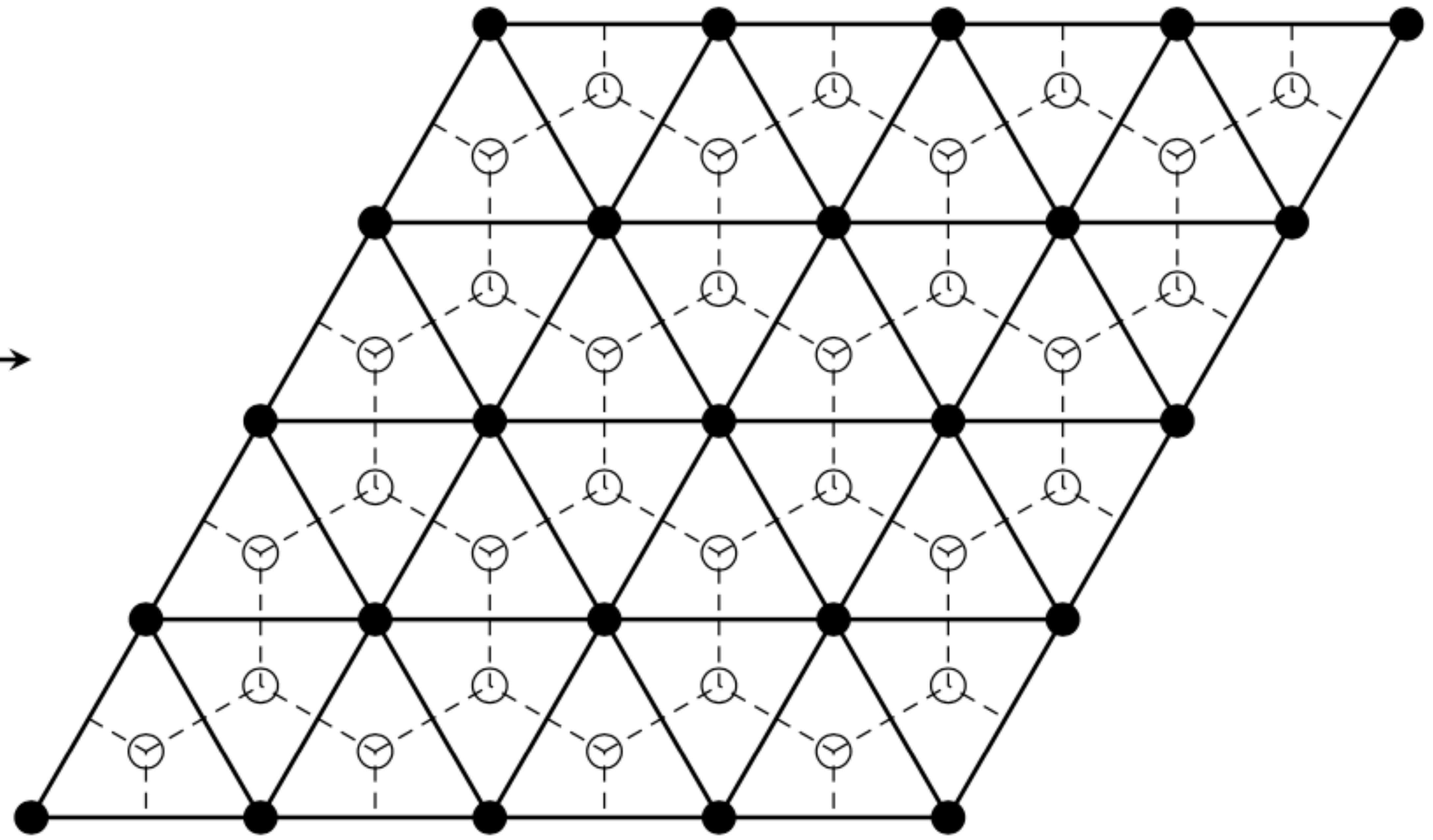
$\lambda_0 = \infty$

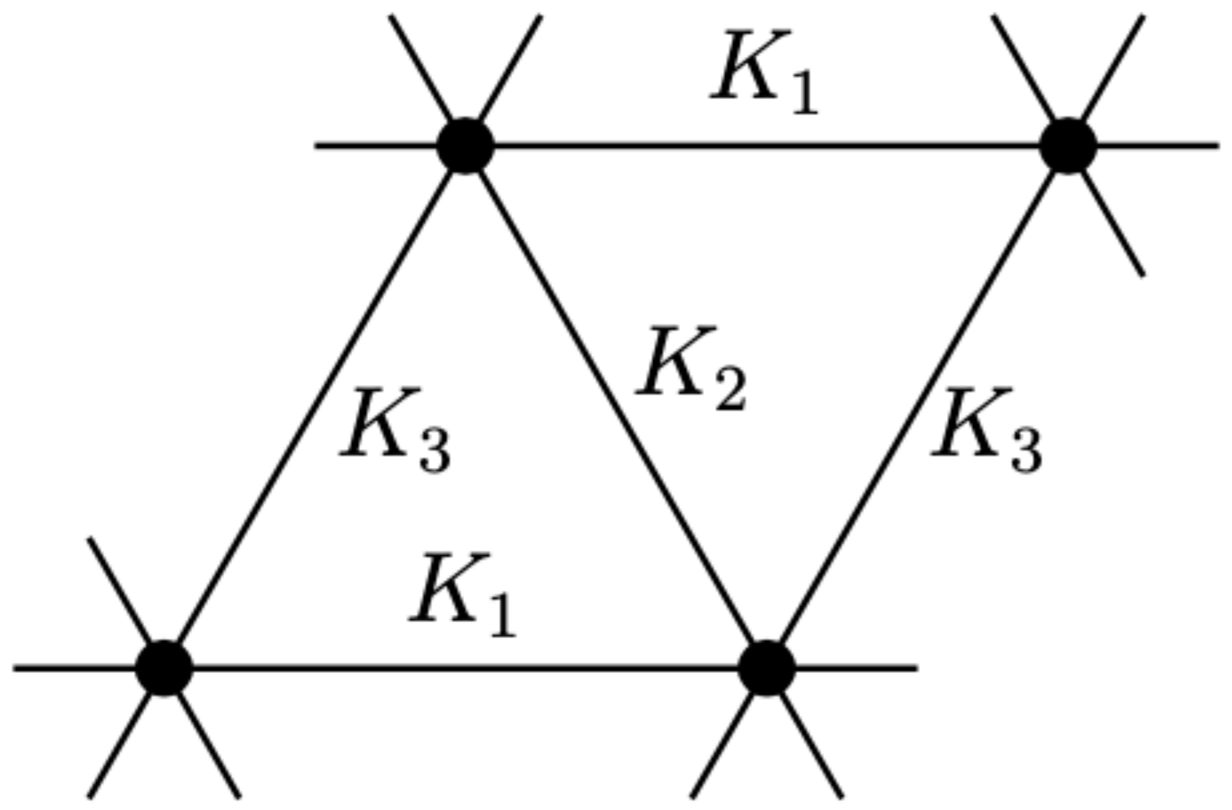
$\lambda_0 = 0$

$$S = \frac{1}{2}[\sum_{\langle i,j \rangle} K_{i,j}(\phi_i - \phi_j)^2 + \lambda_0(\phi_x^2 - 1)^2]$$

Quantum Physic:
Ising Model on the flat Affine Plane

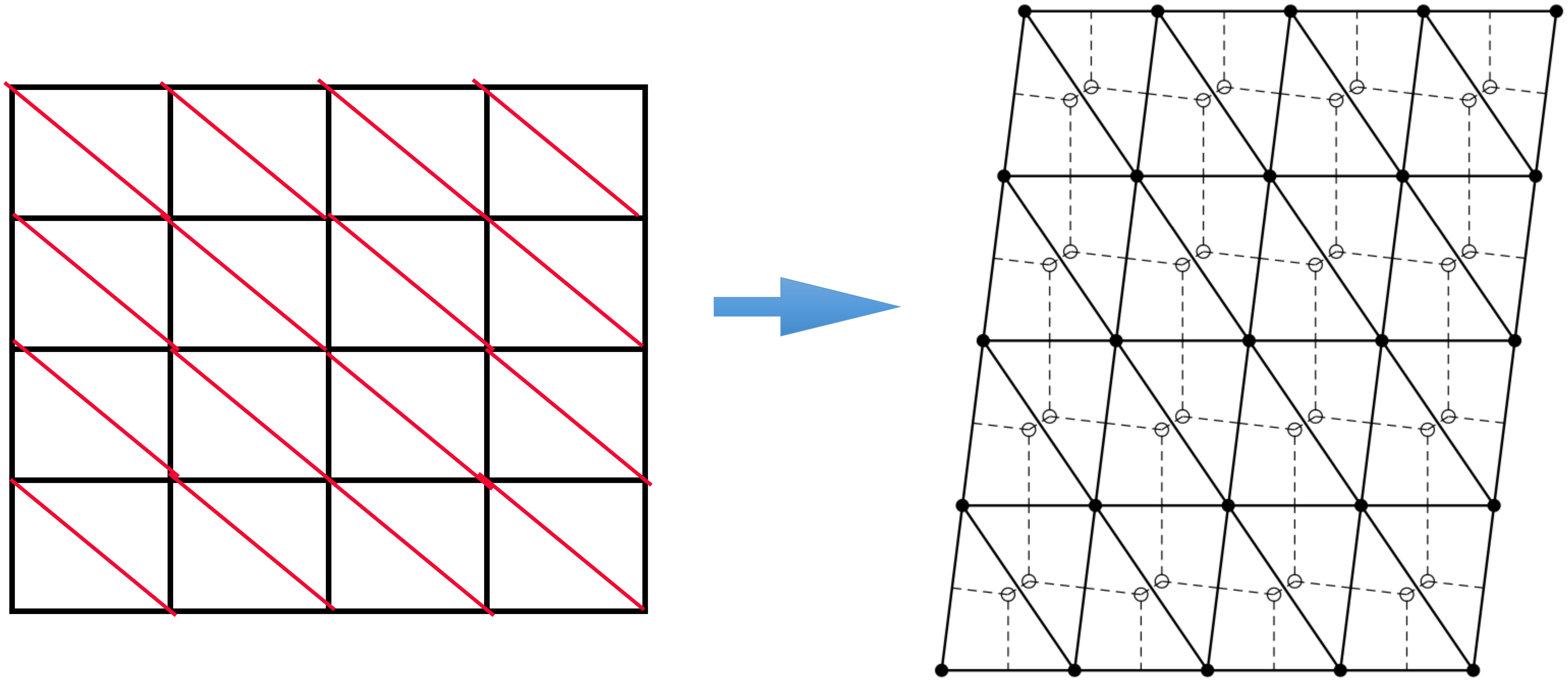






$$Z^{\triangle} = \sum_{s_n=\pm 1} e^{\,K_1 s_n s_{n+\hat{1}} + K_2 s_n s_{n+\hat{2}} + K_3 s_n s_{n+\hat{3}}}\ ,$$

Affine: In 2d flat space
Square to general triangle

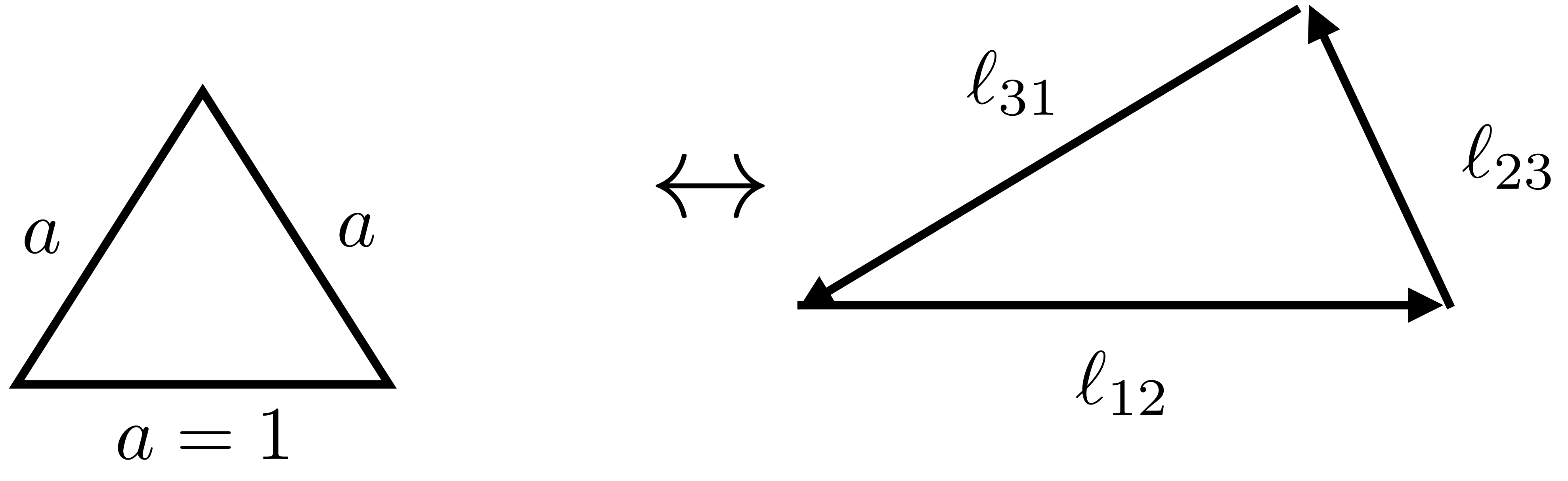


Affine extension of Poincare group:

Unit circle | General ellipse

$$\langle \phi(x,y)\phi(0)\rangle = \frac{1}{(x^2+y^2)^{\Delta_\phi}} \leftrightarrow \frac{1}{(ax^2+bxy+cy^2)^{\Delta_\phi}}$$

Unit equilateral | General simplex/mod Poincare



- d = 2 Poincare: 1 rotation 2 translation
- d = 2 constant metric- 3 parameters: 1 major/minor + 1 orientation + 1 scaling
- General Poincare d(d+1)/2 plus d(d+1)/2 the number of edge in d-simplex - local metric

$$X = A\xi + b \quad \Longrightarrow \quad dx^\mu = A^\mu_i d\xi^i$$

$$ds^2 = d\vec{X}\cdot d\vec{X} = (A^T A)_{ij} d\xi^i d\xi^j = \vec{e}_i \cdot \vec{e}_j d\xi^i d\xi^j = g_{ij} d\xi^i d\xi^j$$

Map of Reggi's Geometry to Lattice Coupling

- Free (FEM) scalar CFT.

$$S_{\text{free}} = \frac{1}{2}\sum_n [K_1(\phi_n - \phi_{n+\hat{1}})^2 + K_2(\phi_n - \phi_{n+\hat{2}})^2 + K_3(\phi_n - \phi_{n+\hat{3}})^2]$$

$$2K_1 = \ell_1^*/\ell_1 \quad , \quad 2K_2 = \ell_2^*/\ell_2 \quad , \quad 2K_3 = \ell_3^*/\ell_3 \ .$$

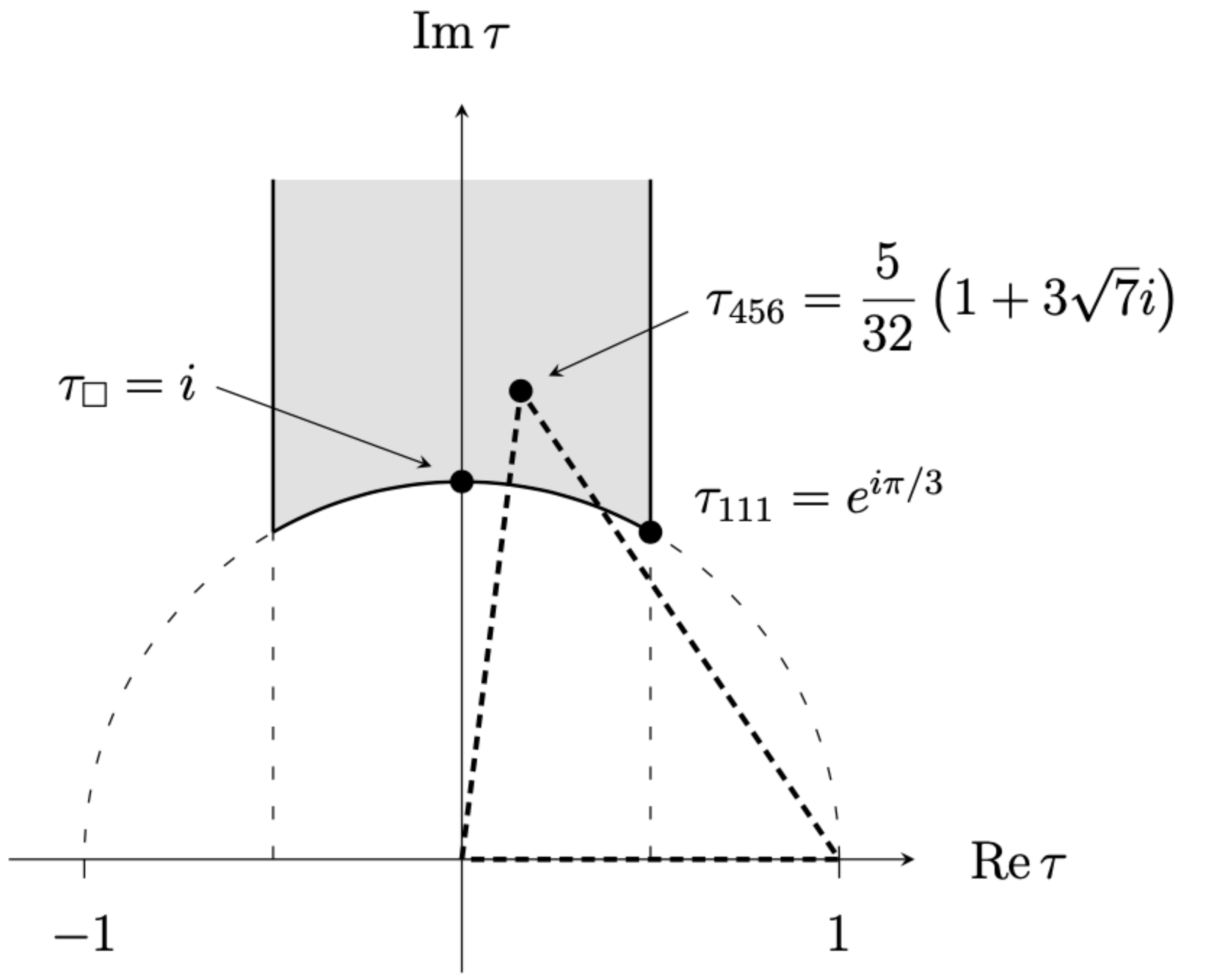
- Exact Ising Quantum Map:

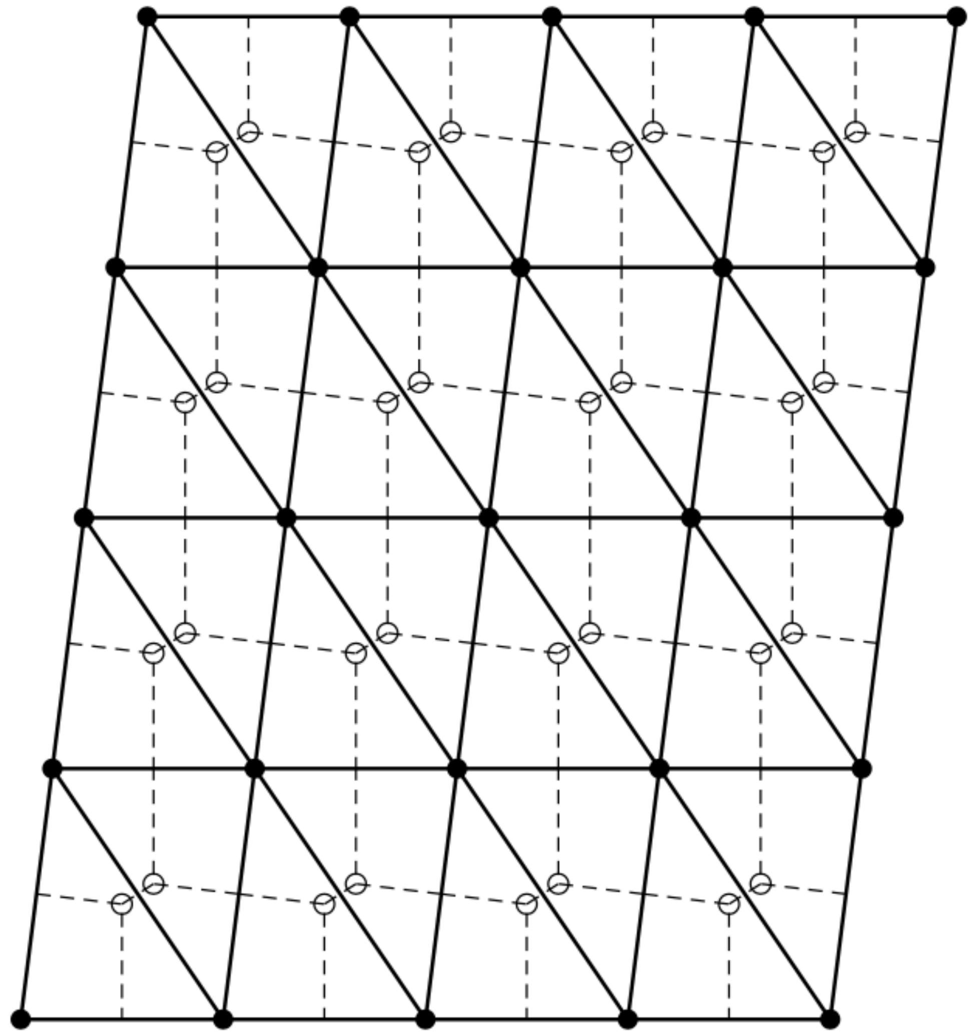
$$\sinh(2K_1) = \ell_1^*/\ell_1 \quad , \quad \sinh(2K_2) = \ell_2^*/\ell_2 \quad , \quad \sinh(2K_3) = \ell_3^*/\ell_3$$

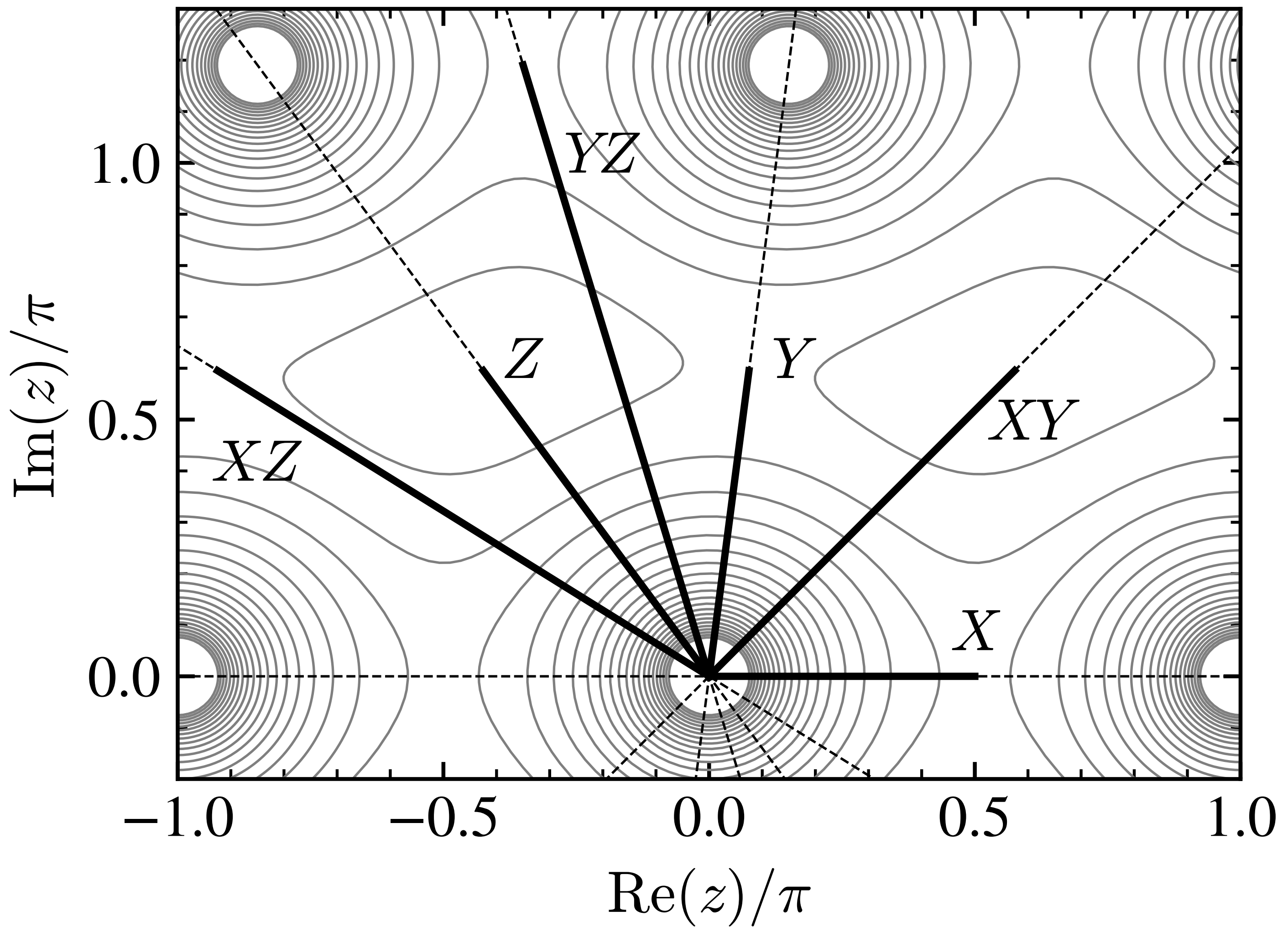
implies critical surface: $p_1p_2 + p_2p_3 + p_3p_1 = 1 \quad \text{with} \quad p_i = \exp(-2K_i)$

- Phi 4th Map? $f(K_i, \lambda_0) = \ell_i^*/\ell_i$

Calculation Modular dependent on the torus

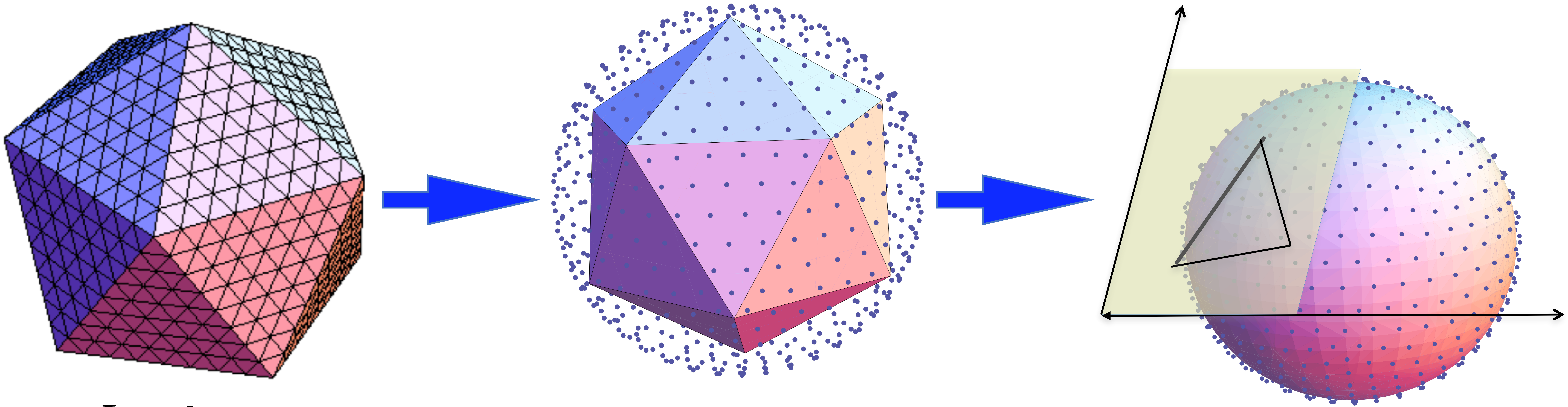






$$\langle \sigma(0)\sigma(z)\rangle = \left|\frac{\vartheta_1'(0|\tau)}{\vartheta_1(z|\tau)}\right|^{1/4} \frac{\sum_{\nu=1}^{4} |\vartheta_\nu(z/2|\tau)|}{\sum_{\nu=2}^{4} |\vartheta_\nu(0|\tau)|}$$

First Attempt (with good results) on refined octahedron



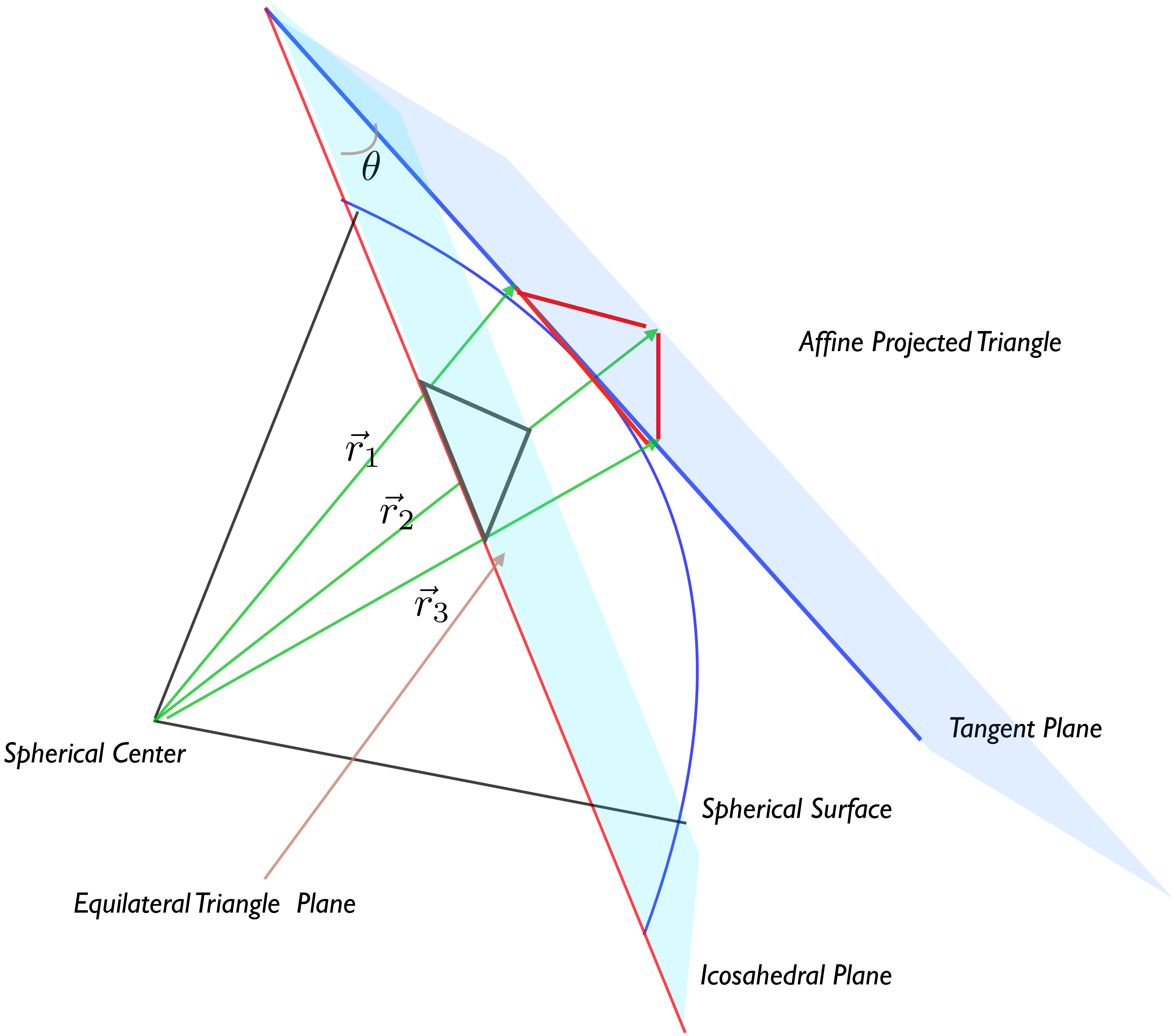
$L = 8$

I = 0 (A),1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of O(3)

$N - F + E = 2$

$$F = N_\triangle = 20L^2 \text{ and dof: } 2N = 4 + 20L^2$$

To O(a^2) the tangent plane is an Affine lattice on each tangent plane.

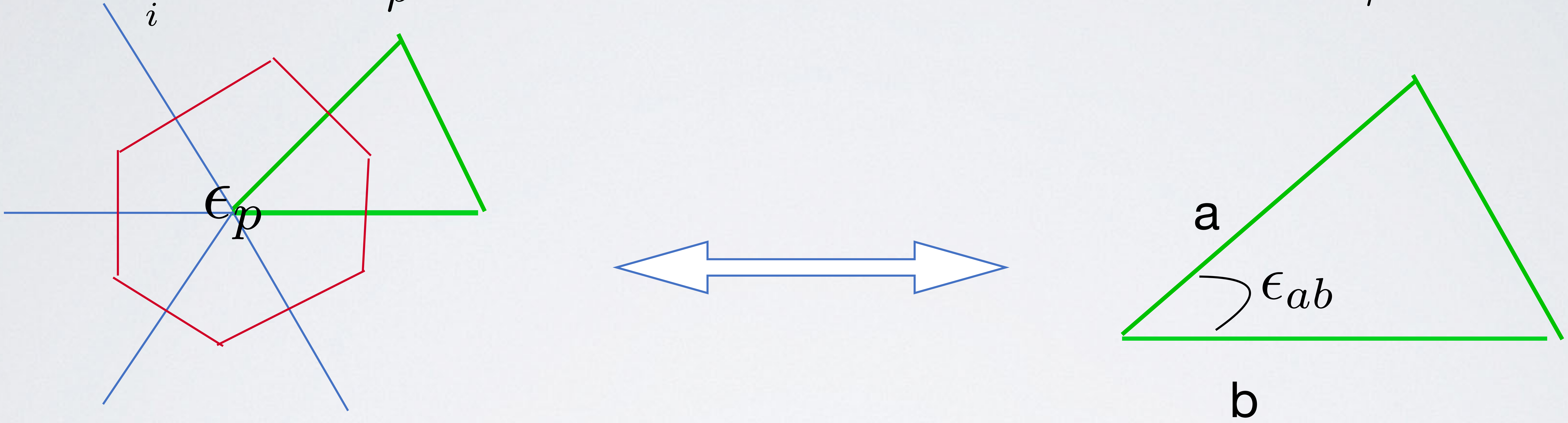


"SMOOTH" SCALAR CURVATURE ON SPHERE

(See 1984 T D LEE et al " Lattice Gravity Near the Continuum")

$$\epsilon_h = \sum_i \epsilon_{i,i+1} = \frac{A_h^*}{\rho^2}[1 + O(\ell^2/\rho^2)]$$

$$\epsilon_{ab} = \theta_{ab}^{sphere} - \theta_{ab}^{flat} = \frac{aa^* + bb^*}{2\rho^2}[1 + O(\ell^2/\rho^2)]$$



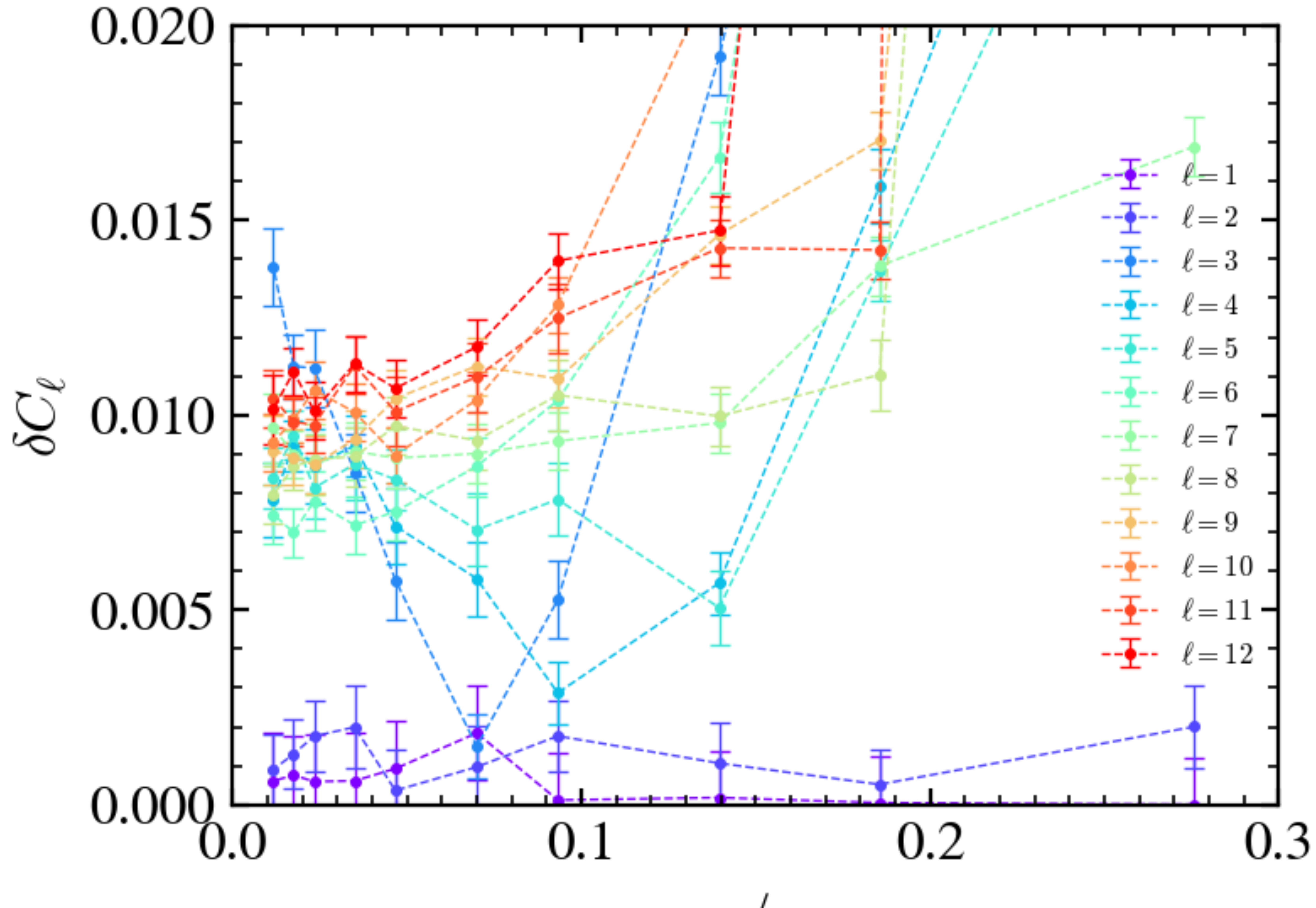
Generalize: Hinge D-2 Volume to closest hybrid to Hinge

$$|\sigma_n \wedge \sigma_n^*| = \frac{(D-n)!n!}{D!}|\sigma_n||\sigma_n^*|$$

$$V_h^{D-2}\epsilon_h = \sum_f \epsilon_{f,f+1} = \frac{D(D-1)}{2}\,\frac{V_{D-2}A_h^*}{\rho^2}[1 + O(\ell^2/\rho^2)]$$

Co-ordinate change gauge to smooth Scalar Curvature

BUT till not right gauge/co-ordinate system

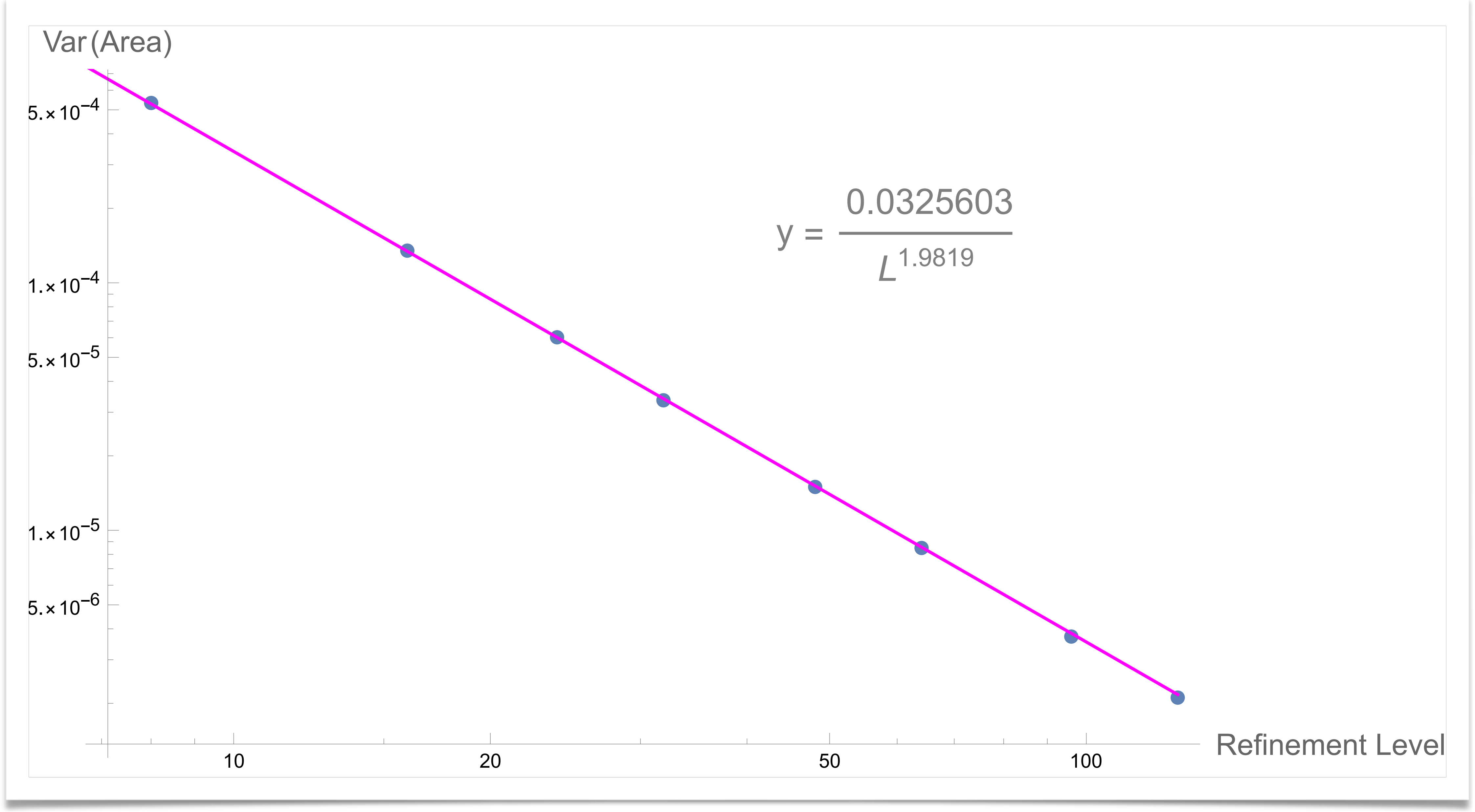


Area Optimization to smooth scalar curvature

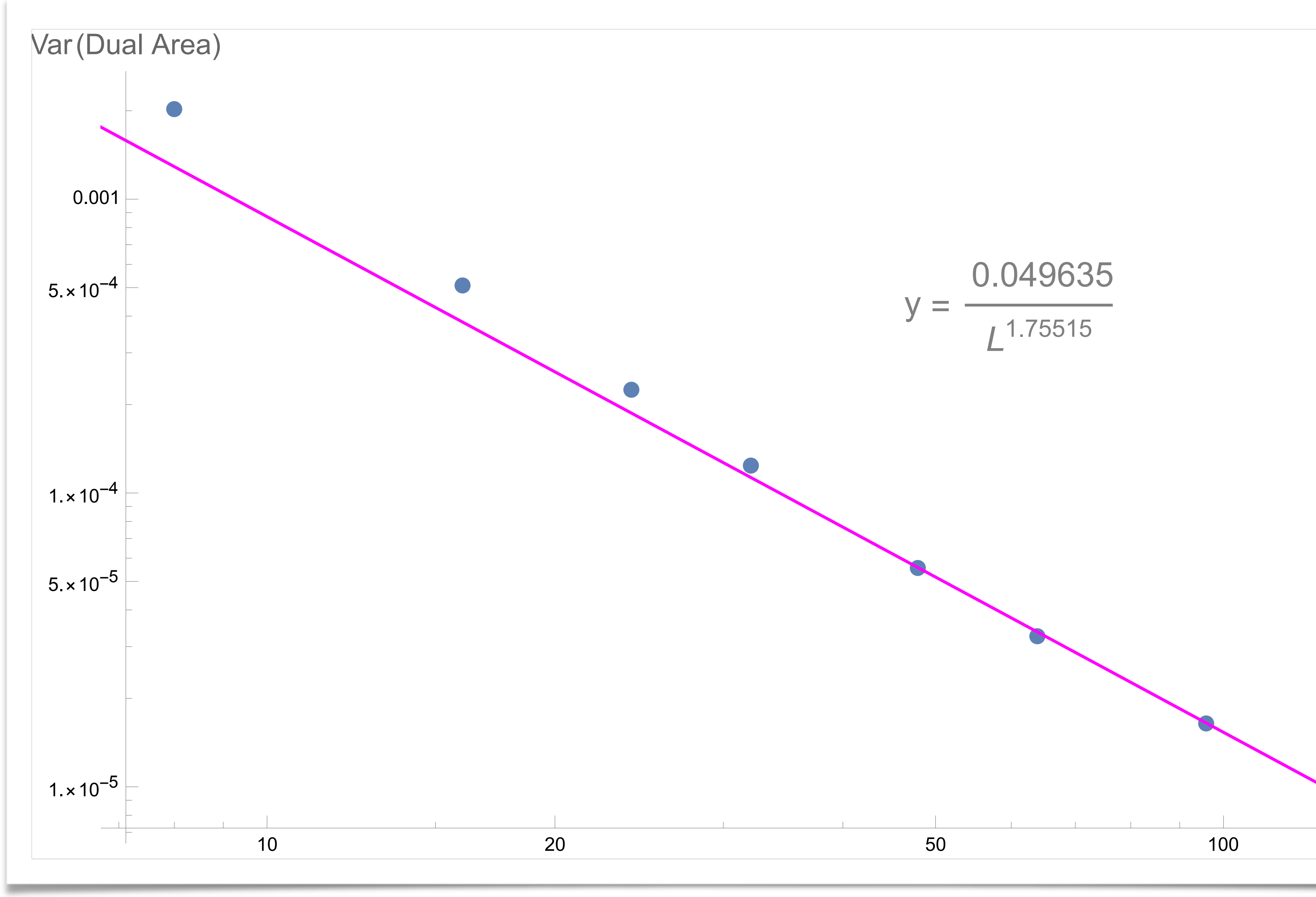
$$S(\ell_{ij}) = N^{-1} \sum_{\triangle} A^2_{\triangle}(l_{ij})$$

dof: $2N = 4 + 20L^3$

Area Variance



Dual Area Variance

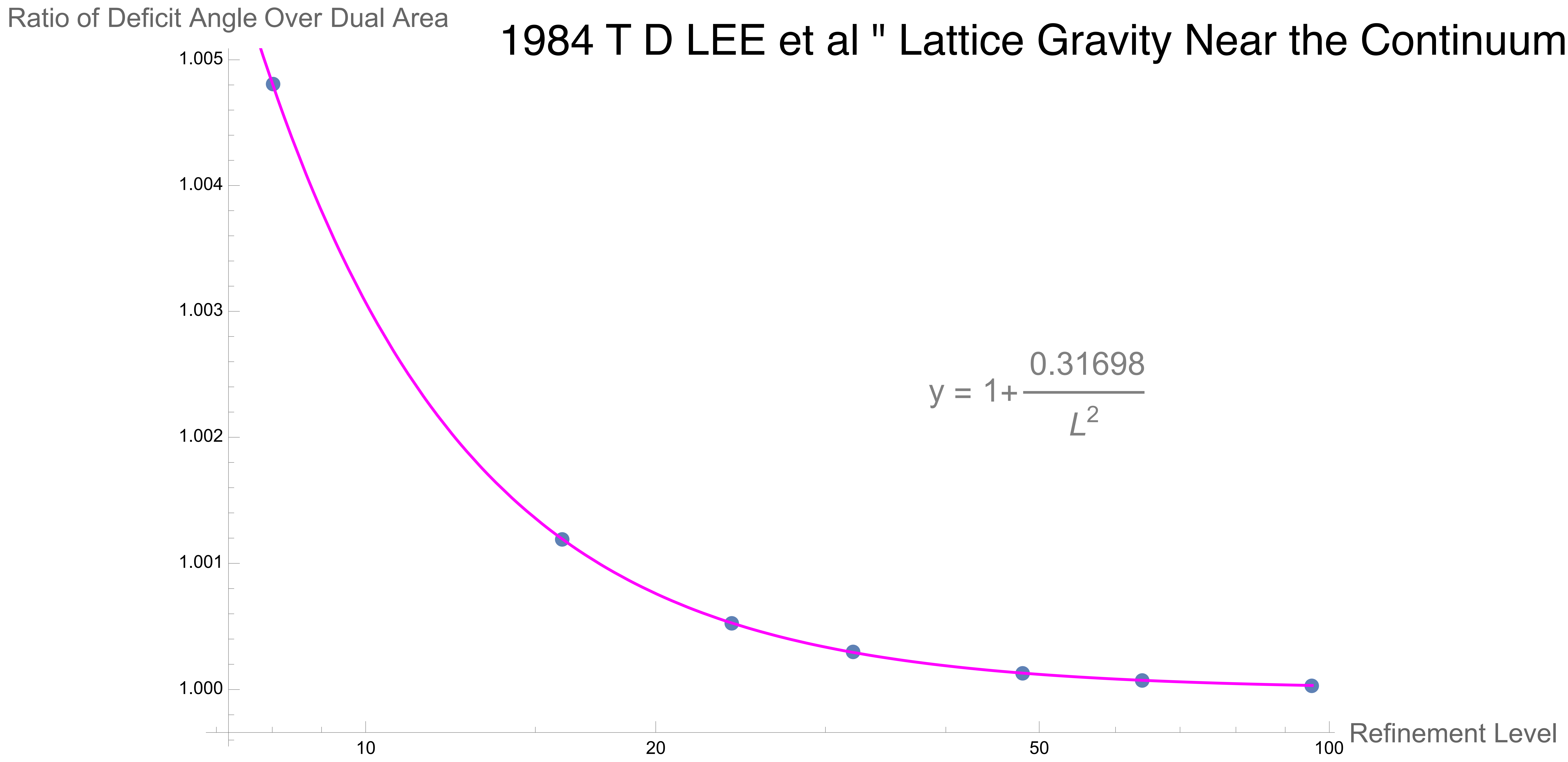


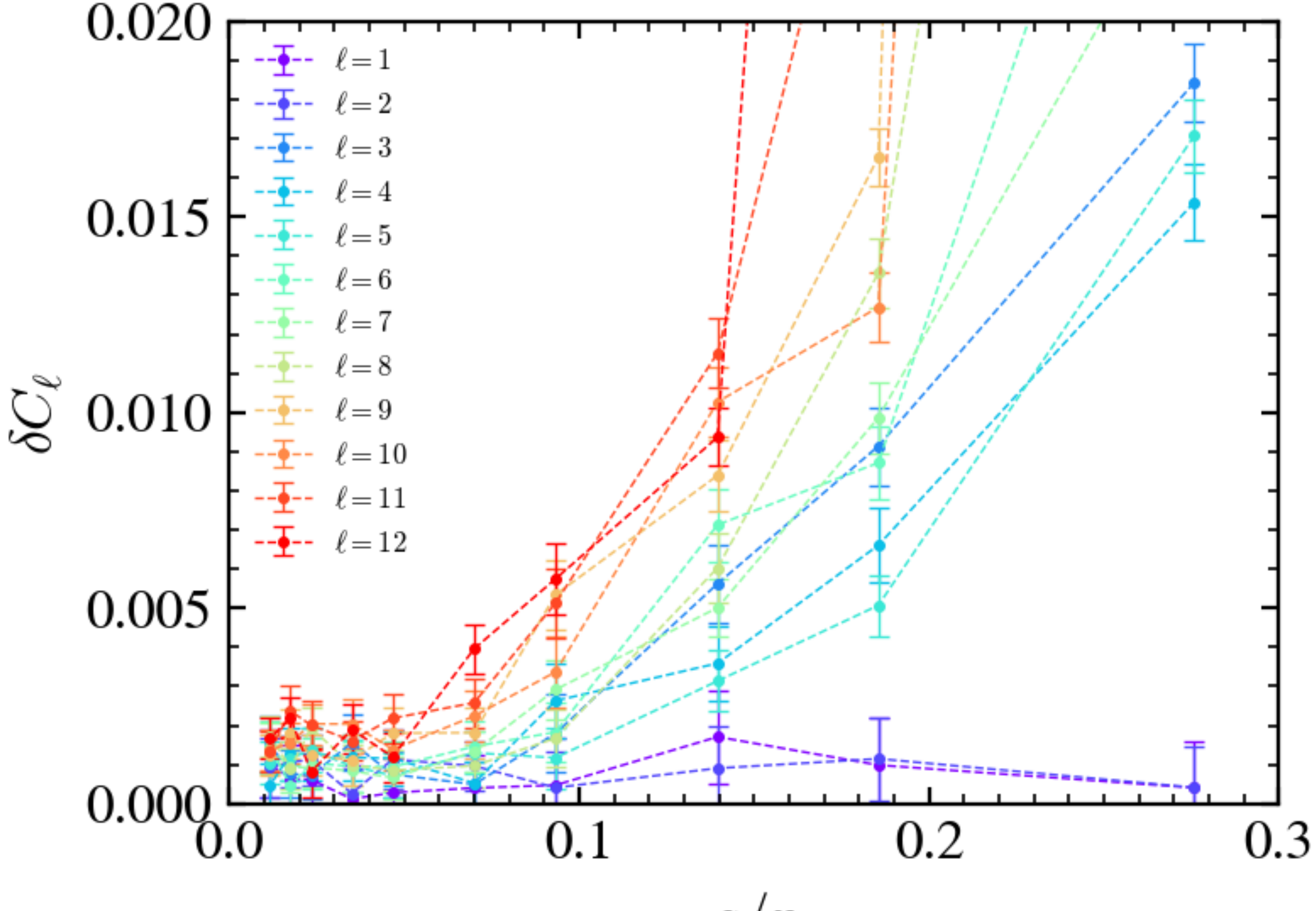
$$4A(a,b,c)^2 = (a+b+c)(-a+b+c)(a-b+c)(a+b-c)$$
$$= a^2b^2c^2/R^2_{\triangle}$$

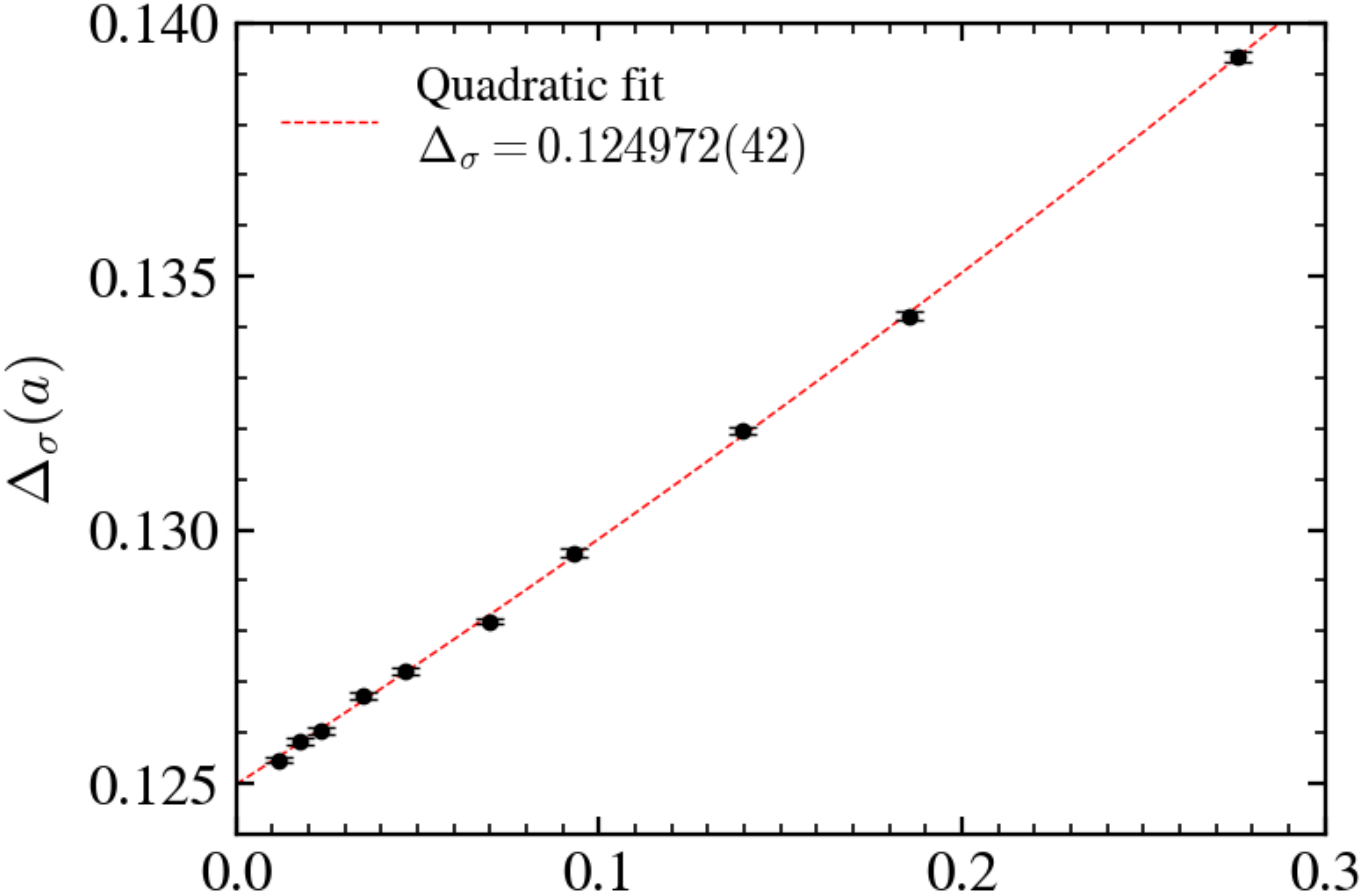
$$a^2 = \ell^2_{12} = |\vec{r}_1 - \vec{r}_2|^2 = 2 - 2\vec{r}_1 \cdot \vec{r}_2$$

Smooth Scalar Curvature Theorem

1984 T D LEE et al " Lattice Gravity Near the Continuum"




0.020
0.015
0.010
0.005
0.000
0.0
0.1
0.2
0.3
δC_ℓ
$\ell = 1$
$\ell = 2$
$\ell = 3$
$\ell = 4$
$\ell = 5$
$\ell = 6$
$\ell = 7$
$\ell = 8$
$\ell = 9$
$\ell = 10$
$\ell = 11$
$\ell = 12$



0.140
0.135
0.130
0.125
$\Delta_\sigma(a)$
0.0
0.1
0.2
0.3
Quadratic fit
$\Delta_\sigma = 0.124972(42)$

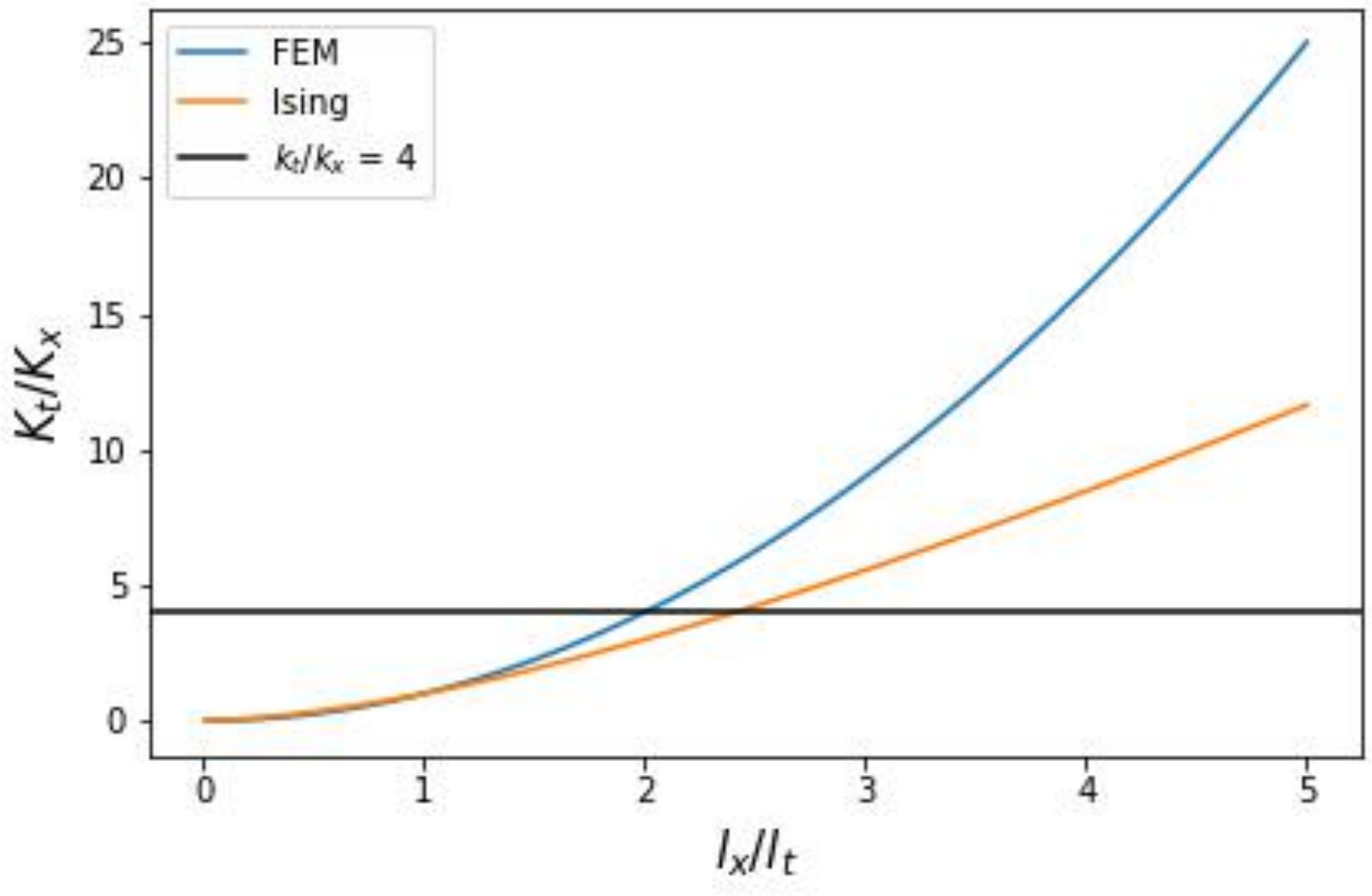
WHAT'S NEXT?

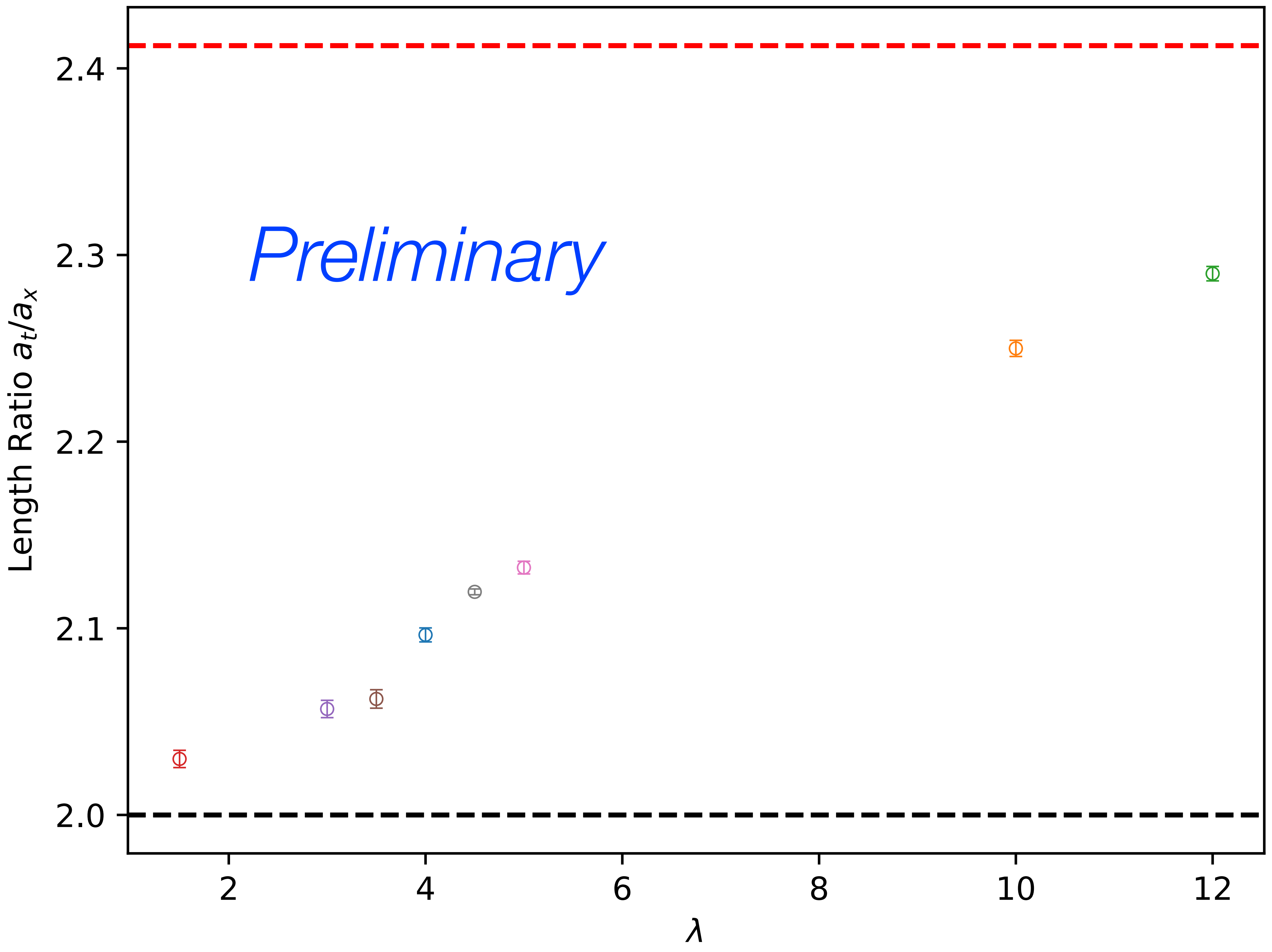
- Precession tests and geometric analysis of Affine map -- e.g. generalized Karsch coefficients and EM tensor.
- Test for 2d phi^4 theory. $f(K_i, \lambda_0) = \ell_i^* / \ell_i$
- 3d Ising model on $\mathbb{S}^3$ & on $\mathbb{R} \times \mathbb{S}^2$
- For 3d an 4d Gauge theories $\mathbb{R} \times \mathbb{S}^3$
  - Put $\mathbb{S}^2$ and $\mathbb{S}^3$ lattice data structures into Grid (with Peter Boyle's help!)
  - Develop general Affine Map adaptive algorithm (aka Machine Learning) for Ricci flow gauge fixing of Co-ordinates.

Find the affine map for 2D ϕ^4

$$S = \frac{1}{2}\sum_x [K_i(\phi_{x+i} - \phi_x)^2 + \lambda_0(\phi_x^2 - 1)^2]$$

$$f(K_i, \lambda_0) = \ell_i^* / \ell_i$$





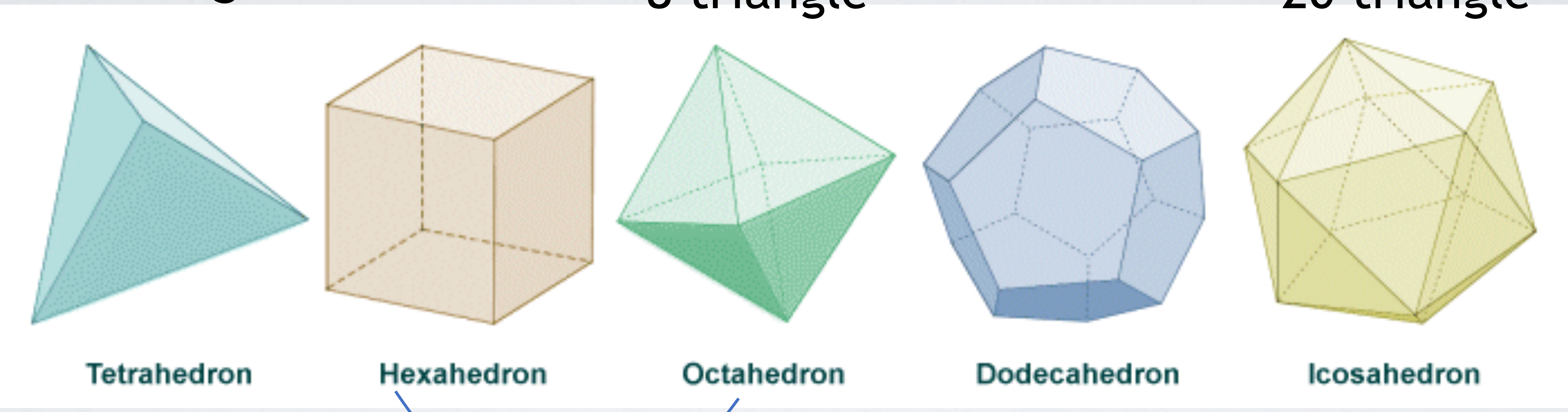
$$f(K_i, 0) = K_i \quad \leftrightarrow \quad f(K_i, \infty) = \sinh(2K_i)$$

2D & 3D SIMPLCIAL PLATONIC SOLIDS

4 triangle

8 triangle

20 triangle



dual

self dual

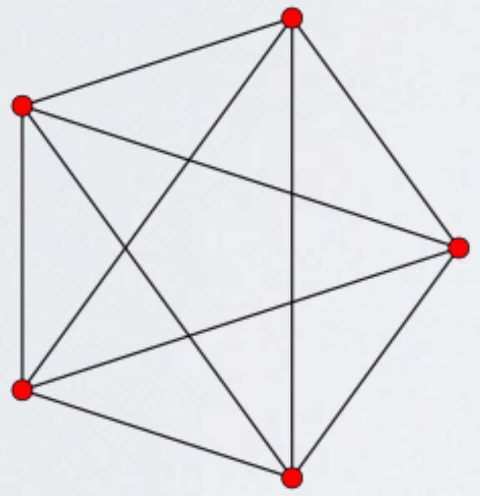
5 tetra

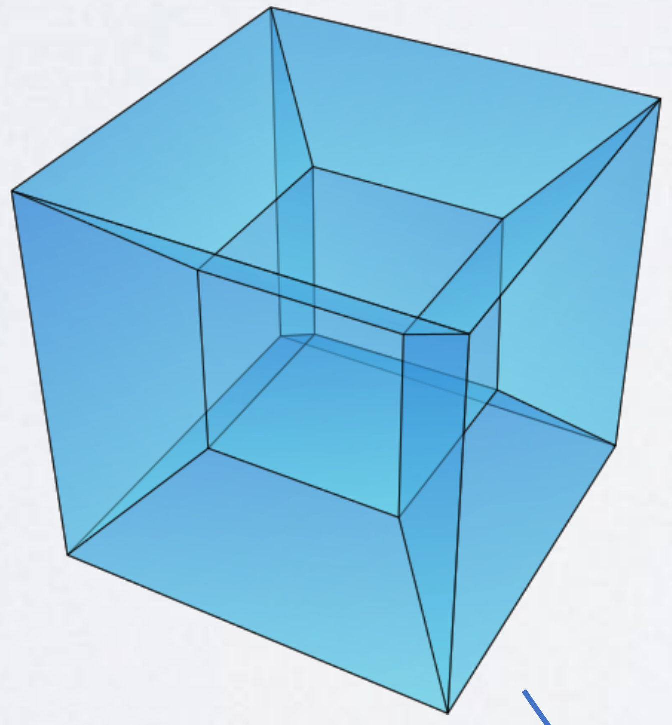
8 cubes

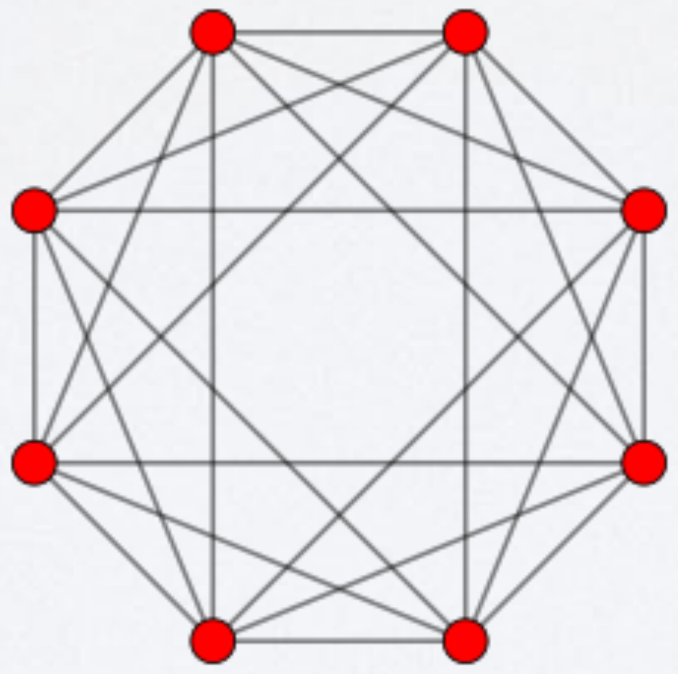
16 tetra

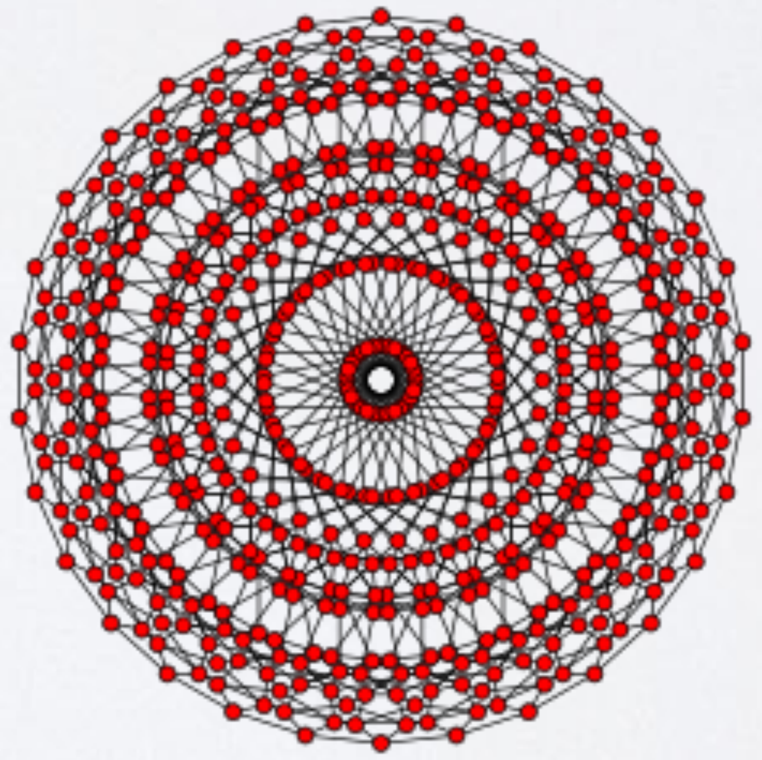
120 dedaca

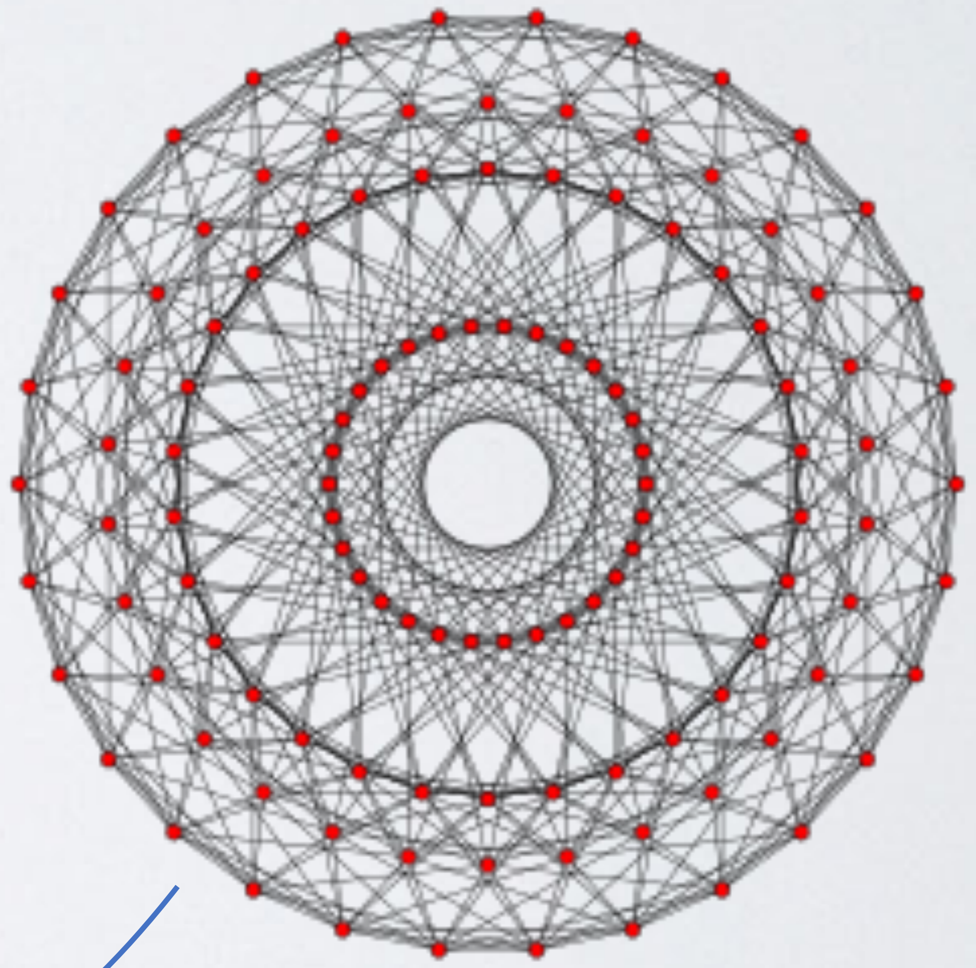
600 tetra











self dual

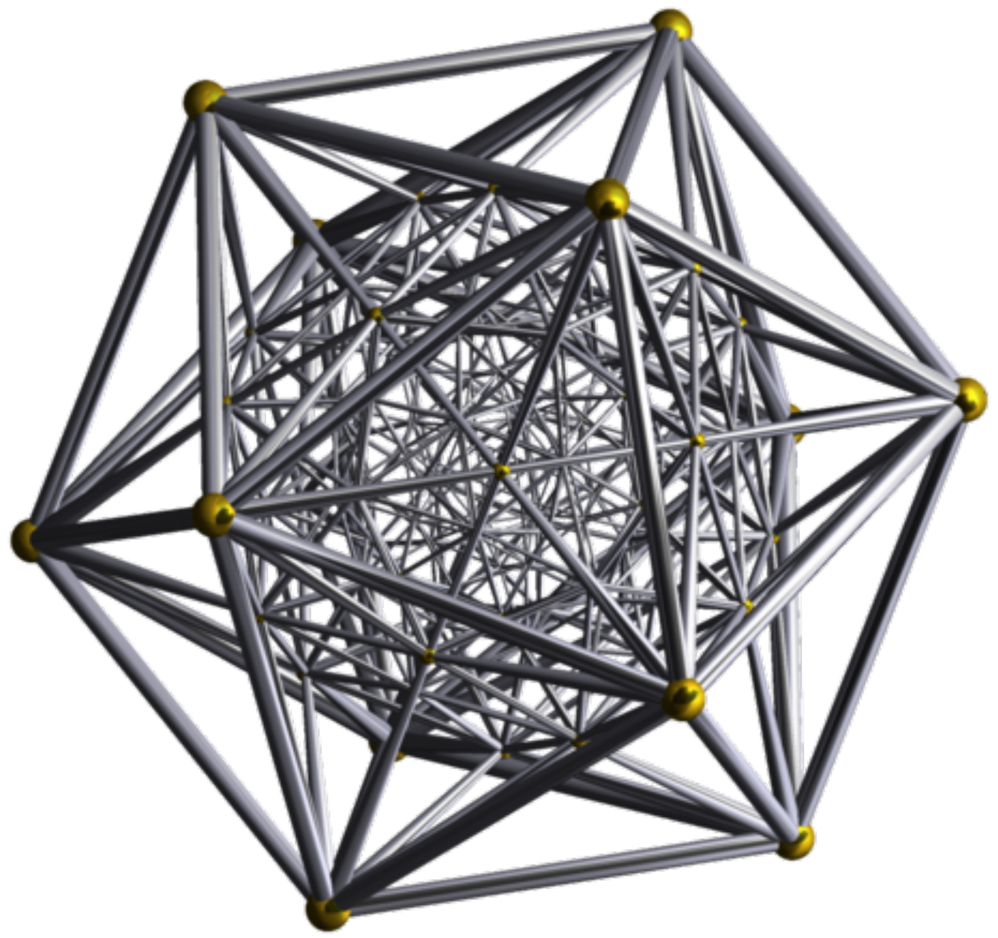
6th self dual with 24 octahedrons

Euler N - E + F - V = 0

https://en.wikipedia.org/wiki/Regular_4-polytope#

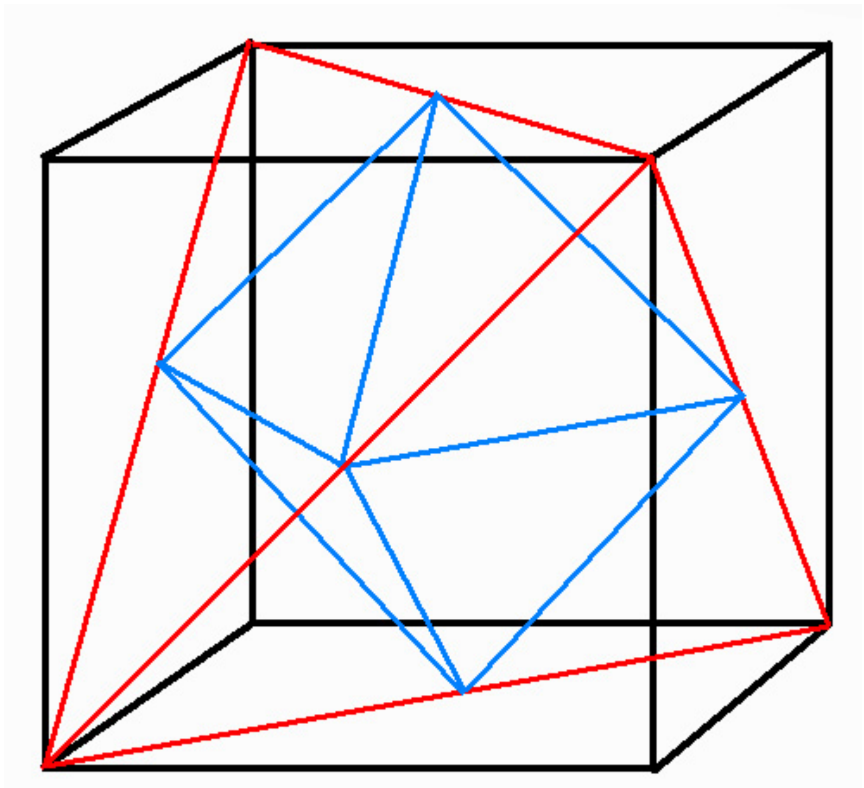
3 Spheres and 4D Radial Simplicial Lattices

$$\mathbb{S}^3 \quad \Longrightarrow \quad \mathbb{R} \times \mathbb{S}^3$$



Aristotle' s 2% Error!

$$(2\pi - 5ArcCos[1/3])/(2\pi) = 0.0204336$$



Fast Code Domains of
Regular 3D Grids on Refinement

600 cell: "Square of the icosahedron" –Symmetries 1440= 120 * 120 the 120 copies of icosahedron

$$O(4) \sim SU(2) \times SU(2)$$

The full symmetry group of the 600-cell is the Weyl group of H_4. This is a group of order 14400. It consists of 7200 rotations and 7200 rotation-reflections. The rotations form an invariant subgroup of the full symmetry group.

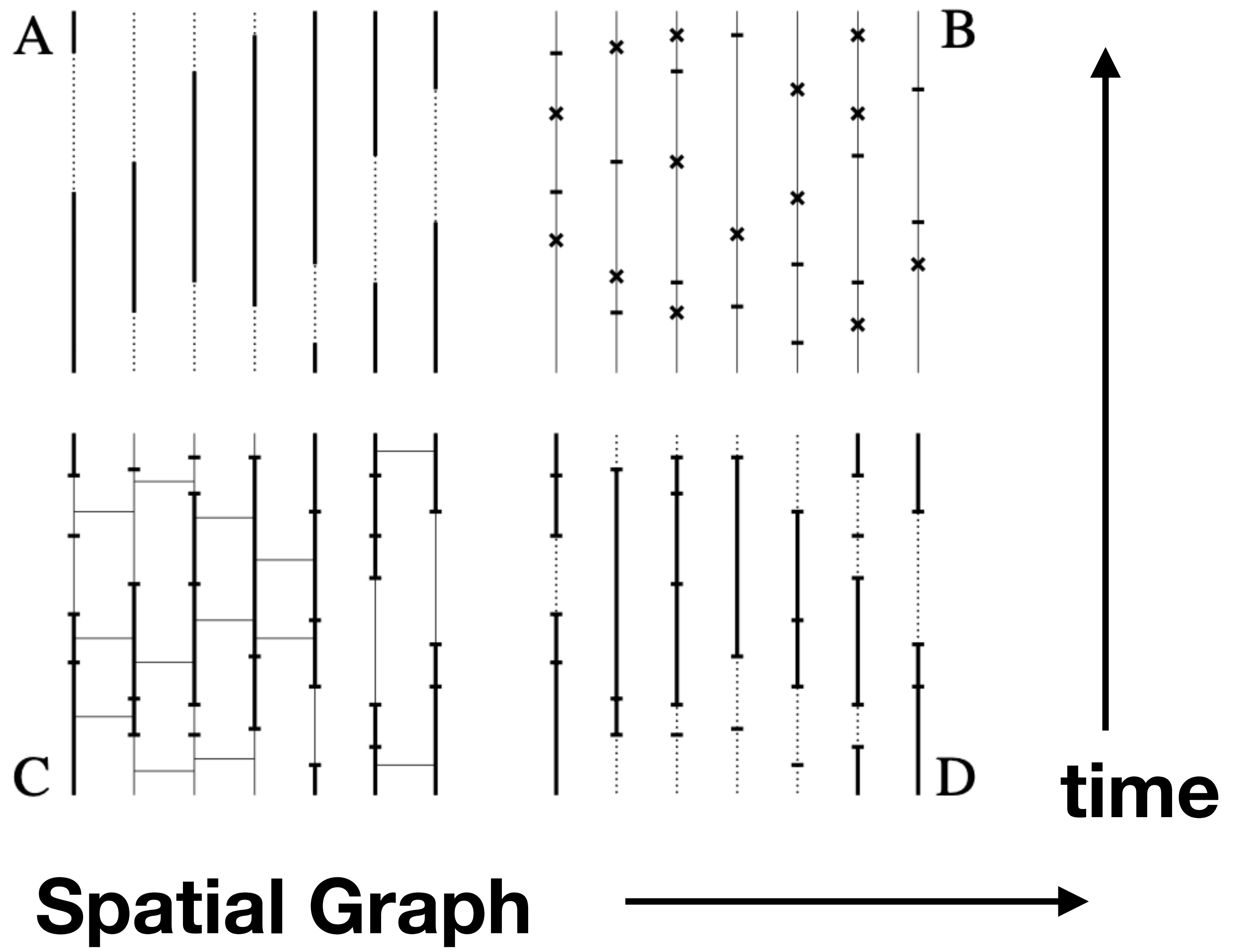
*Continuous Time (aka Euclidian Hamiltonian) Cluster Monte Carlo**

Affine: $a_t \to 0$

$$S = -\sum_{t,i} K_i^0 s_{t,i} s_{t+1,i} - \sum_{\langle i,j \rangle} K_{ij}^{\perp} s_{t,i} s_{t,i}$$

$$H = \sum_i \Gamma_i \sigma_i^x + \sum_{\langle i,j \rangle} \widetilde{K}_{ij} \sigma_i^z \sigma_j^z$$

$$K_{ij} = a_t \widetilde{K}_{ij} \quad , \quad e^{-2a_t K_i^0} = \tanh(a_t \Gamma_i)$$



- A state space (real value decay times)
- B Poisson Decays: $$P(t) = \Gamma e^{-\Gamma t}$$
- C. Spatial Percolation $$P_{ij} = 1 - e^{-2\Delta t_{overlap} \widetilde{K}_{ij}}$$
- D. SW Flip clusters for new state A

Pretty Easy to Program with Connected Components Graph algorithms:
Works for 1 + d Radial Quantization (Sphere) Ising/SUSY/Warped AdS etc

***See: 1998: H. Rieger, N. Kawashima**
Application of a continous time cluster algorithm to the Two-dimensional Random Quantum Ising Ferromagnet

Duality: $\cosh(2a_t K_i^0)\cosh(a_t \Gamma_i) = 1$


THANKS --
See Y'all at George's Poster
1
2
0
simplex
FERMILAB-POSTER-24-0170-T
Matching Curved Lattices to Anisotropic Tangent Planes
George T. Fleming, Theory Division, Fermilab
with R. Brower (Boston U.) J. Lin (Carnegie-Mellon U.), N. Matsumoto (Boston U.)
Near-Conformal Field Theories
• Strongly-coupled near-conformal field theories could be important for BSM physics.
• Example: composite Higgs boson $H \sim \bar{Q}Q$, $v \sim \langle \bar{Q}Q \rangle$.
• This implies a composite Yukawa mechanism to give mass to SM fermions: $y_f \langle \bar{Q}Q \rangle \bar{f}f / \Lambda^2$
• But, this also leads to flavor changing neutral currents $(\bar{f}f)(\bar{f}f)/\Lambda^2$ which requires $\Lambda > 1000\ TeV$.
• So, composite Higgs theory must be strongly-coupled over a range of 0.1 – 1000 TeV.
• Very hard to study on hypercubic flat torus. See talks by A. Hasenfratz and O. Witzel on Friday.
Radial Quantization
• Eigenstates of Dilatation operator defined on surfaces of constant radius.
• Eigenstates labeled by angular momenta (ℓ, m_ℓ) due to rotational invariance.
• Dynamical dispersion relation (conformal): $\Delta_{O,\ell} = \Delta_{O,0} + \ell$
• Correlations (conformal): $C(\ell, t, t') = \sum_O B(\Delta_O, \ell)\, e^{-\Delta_{O,\ell}|t-t'|}$
• Near-conformal would modify integer spacing and t-dependence.
• Challenge: How to define action on irregular spherical lattice that has rotational symmetry in continuum limit?
Quantum Finite Elements
• Limited Solution: Finite Element Method (FEM) gives classically perfect action. QFE adds perturbative counterterms.
• Method worked for critical 3D ϕ^4 theory but very slow convergence to continuum limit.
• Also, discovered a novel coupling to local curvature density, $Ric(x)\phi^2(x)$, which further slowed convergence $\sim O(a^{0.41})$.
• Lesson 1: Adjust lattice so curvature density is uniform a la Regge calculus.
• Lesson 2: Need a better method to define lattice action which is closer to strongly coupled IR fixed point.
Beyond QFE: Affine Conjecture
See Talk by R. Brower on Tues.
• Start with uniform simplicial graph on a refined regular (D+1)-polytope (e.g. Icosahedron or 600-cell)
• Project vertices to $\mathbb{S}^D$. Optimize vertex positions to uniform curvature density (Regge calculus) while preserving graph structure and isometries of polytope.
• Each D-simplex no longer has uniform edge lengths but still defines a "tangent" plane.
• Tesselate each tangent plane with an asymmetric simplicial honeycomb (A_D root lattice) using edge lengths of associated D-simplex.
• Challenge: In the tangent plane, find the anisotropic bare lattice action that dynamically produces the desired ratios of edge lengths.
• That tangent plane action is the lattice action for the associated D-simplex on the $\mathbb{S}^D$.
• Proof of principle: critical Ising model on $\mathbb{S}^2$, E. Owen and R. Brower, 2023.
Specific Goal for This Work
• Solve the critical D=3 Ising model on a general anisotropic face-centered cubic (FCC, aka A_3root lattice).
• The isotropic FCC case has been solved many times: P.H.Lundow et al 2009, U. Yu 2015.
• Under the affine conjecture, a general solution would enable critical Ising model calculations on discretized $\mathbb{S}^3$ starting from 600-cell (higher-dimensional icosahedron) and tessellating each regular tetrahedral cell with an FCC lattice.
• Note a general anisotropic FCC lattice has 6 unique lengths and any lattice can be transformed to the isotropic FCC lattice by affine transformation which also has 6 free parameters.
Generalized FCC Ising model
• Affine FCC partition function
$Z(K_1, \cdots, K_6) = \sum_{s_n = \pm 1} e^{K_1 s_n s_{n+\hat{1}} + \cdots + K_6 s_n s_{n+\hat{6}}}$,
$E_{n,\hat{e}} = -s_n s_{n+\hat{e}}$
• Multihistogram master equation (solved iteratively)
$Z_k = \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{1}{\sum_{j=1}^{R} N_j Z_j^{-1} e^{(\vec{K}_k - \vec{K}_j)\cdot \vec{E}_{r,i}}}$
where R is number of runs, N_r is length of run r, $\vec{E}_{r,i}$ are energies on i-th configuration of run r.
• Observables for any other nearby $\vec{K}$:
$\langle \mathcal{O}(\vec{K}) \rangle = \frac{1}{Z(\vec{K})} \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{\mathcal{O}_{r,i}}{\sum_{j=1}^{R} N_j Z_j^{-1} e^{(\vec{K} - \vec{K}_j)\cdot \vec{E}_{r,i}}}$
$Z(\vec{K}) = \sum_{r=1}^{R} \sum_{i=1}^{N_r} \frac{1}{\sum_{j=1}^{R} N_j Z_j^{-1} e^{(\vec{K} - \vec{K}_j)\cdot \vec{E}_{r,i}}}$
First Test
• First test: $\frac{K_2}{K_1} = \frac{K_3}{K_1} = 1$, $\frac{K_4}{K_1} = \frac{K_5}{K_1} = \frac{K_6}{K_1} \in \{0.94, 0.97, 1.00, 1.03, 1.06\}$
• K_1 is tuned close to critical point.
• Solve multihistogram consistency condition for all 35 runs, each run $N_r = 50{,}000$ configs.
$V = 128^3$
$V = 128^3$
$V = 128^3$
• Important: all susceptibilities peak at same critical coupling.
• Using multihistogram reweighting, find critical surface $\vec{K}_{crit}$ by identifying peak in $Cov(\vec{E}, \vec{E})$. In general, it is a 5-d surface with permutation symmetry.
• Then, along critical surface compute two-point function:
$\langle s(\vec{x}) s(0) \rangle = \frac{1}{(x_i\, G_{ij}(\vec{K}_{crit})\, x_j)^{\Delta_\sigma}}$
References for Further Reading
Brower et al, PRD 104 (2021) 094502.
Glück et al, PRD 109 (2024) 114518.
Brower and Owen, PRD 108 (2023) 014511.
Brower and Owen, arXiv:2407.00459 [hep-lat].
Ferrenberg-Swendsen, PRL 63 (1989) 1195.
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Fermi National Accelerator Laboratory
Fermilab
U.S. DEPARTMENT OF ENERGY

BACK UP

First step: Construct the Classical Simplicial Theory

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d x \sqrt{g(x)}\left[g^{\mu\nu}(x)\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi^2(x) + \lambda\phi^4(x)\right]$$

$g_{\mu\nu}(x)$

Quantum field $\phi(x)$

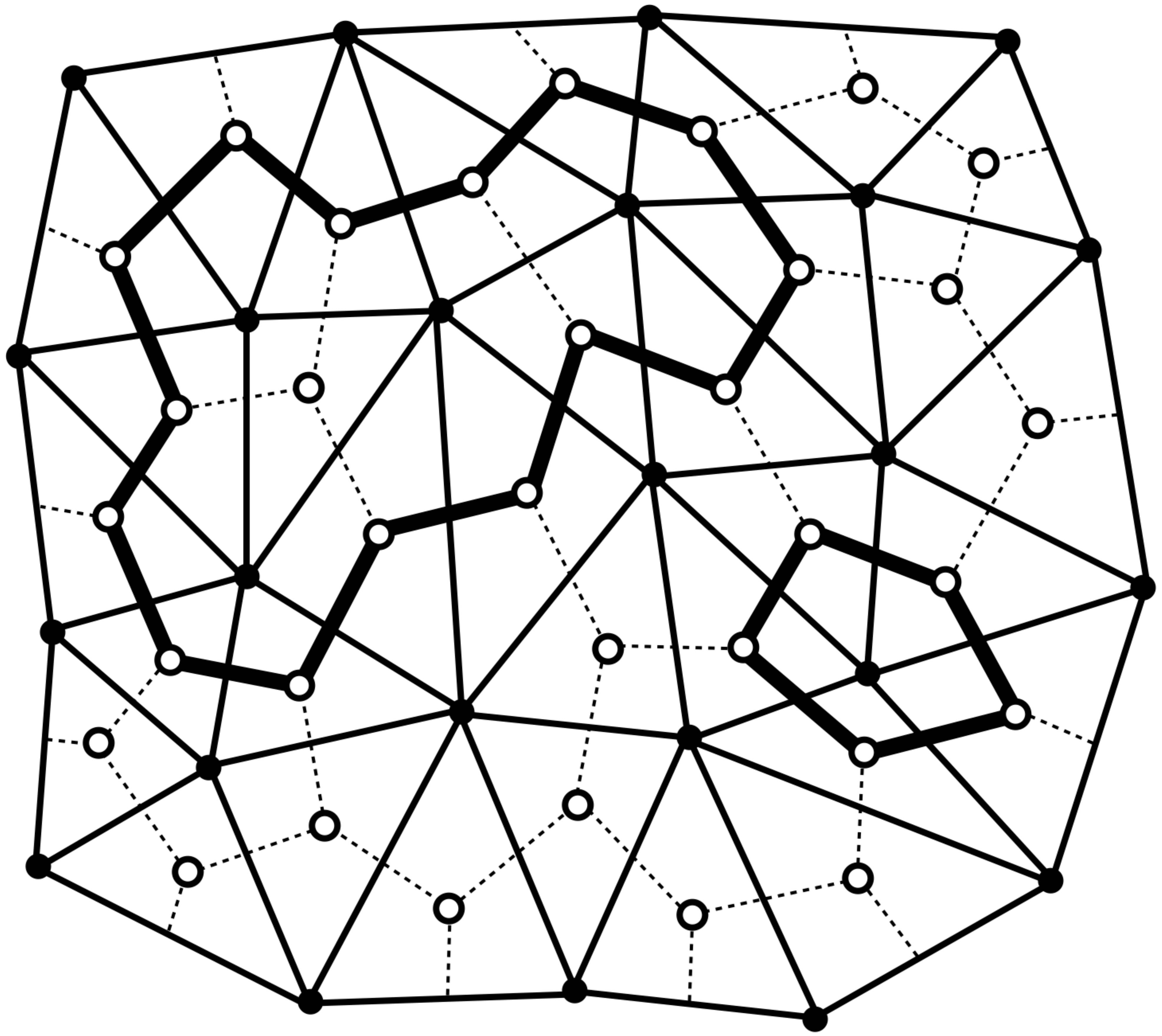
Regge Calc Geometry

Finite Element Method

Classical Simplicial Action

$$S_{FEM} = \frac{1}{2}\Big[\sum_{y\in\langle x,y\rangle} K_{xy}(\phi_x - \phi_y)^2 + \sqrt{g_x}[\xi\ Ric\ \phi_x^2 + m_0^2\phi_x^2 + \lambda_0\phi_{t,x}^4]\Big]$$

3 Equivalent Loop Expansion for Partition Functions!



$$S_\triangle = -\sum_{\langle ij \rangle} K_{ij} s_i s_j$$

$$S_{\rm dual} = -\sum_{\langle ij \rangle} L_{ij} s_i s_j$$

$$S_\psi = \frac{1}{2}\sum_i \bar\psi_i \psi_i - \sum_{\langle ij \rangle} \kappa_{ij} \bar\psi_i P_{ij} \psi_j \ ,$$

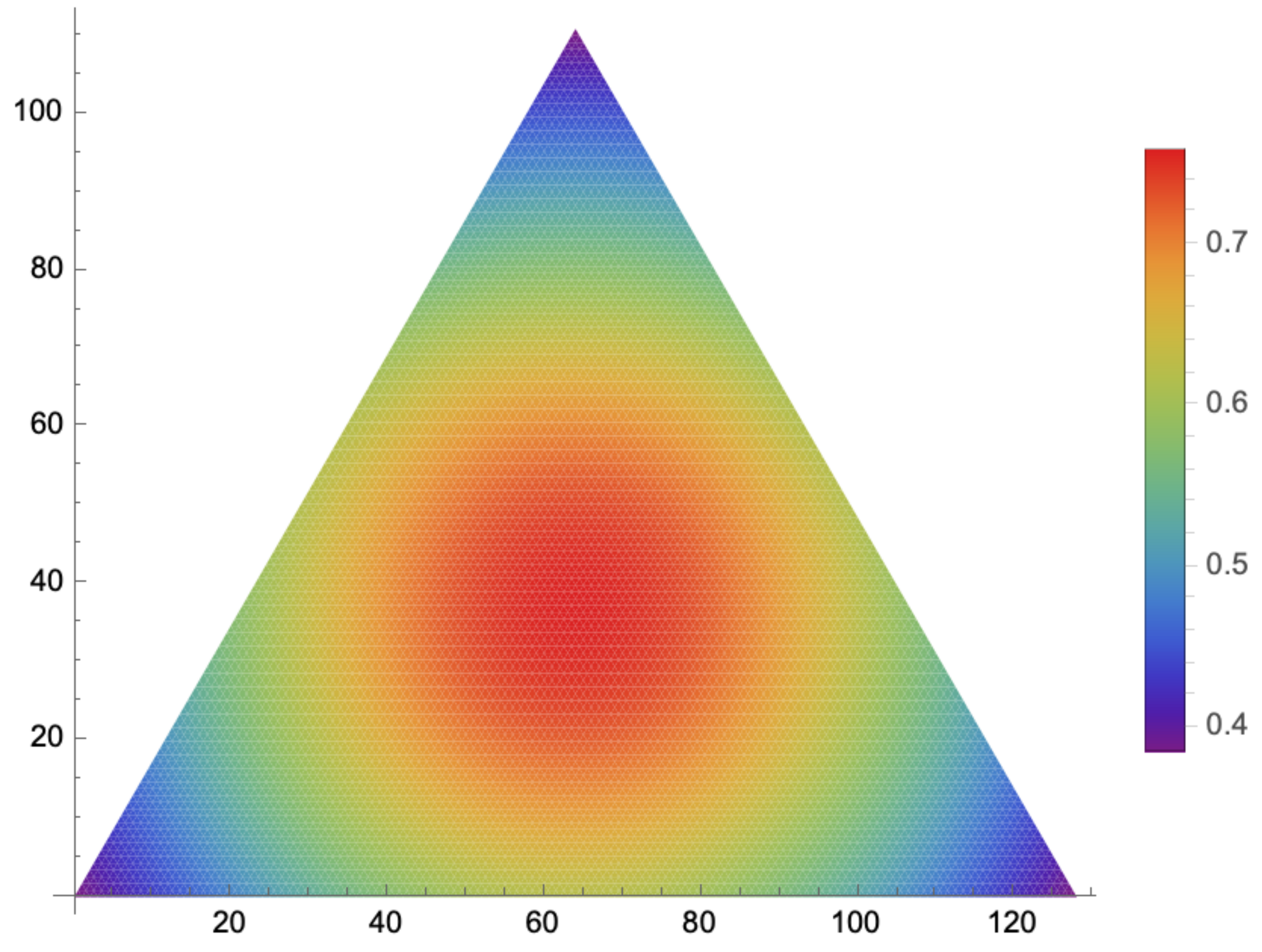
Kramers Wannier High T/Low T Loop expansion Wilson-Majorana Lattice Fermions

$$\sinh 2K_{ij} \sinh 2L_{ij} = 1$$

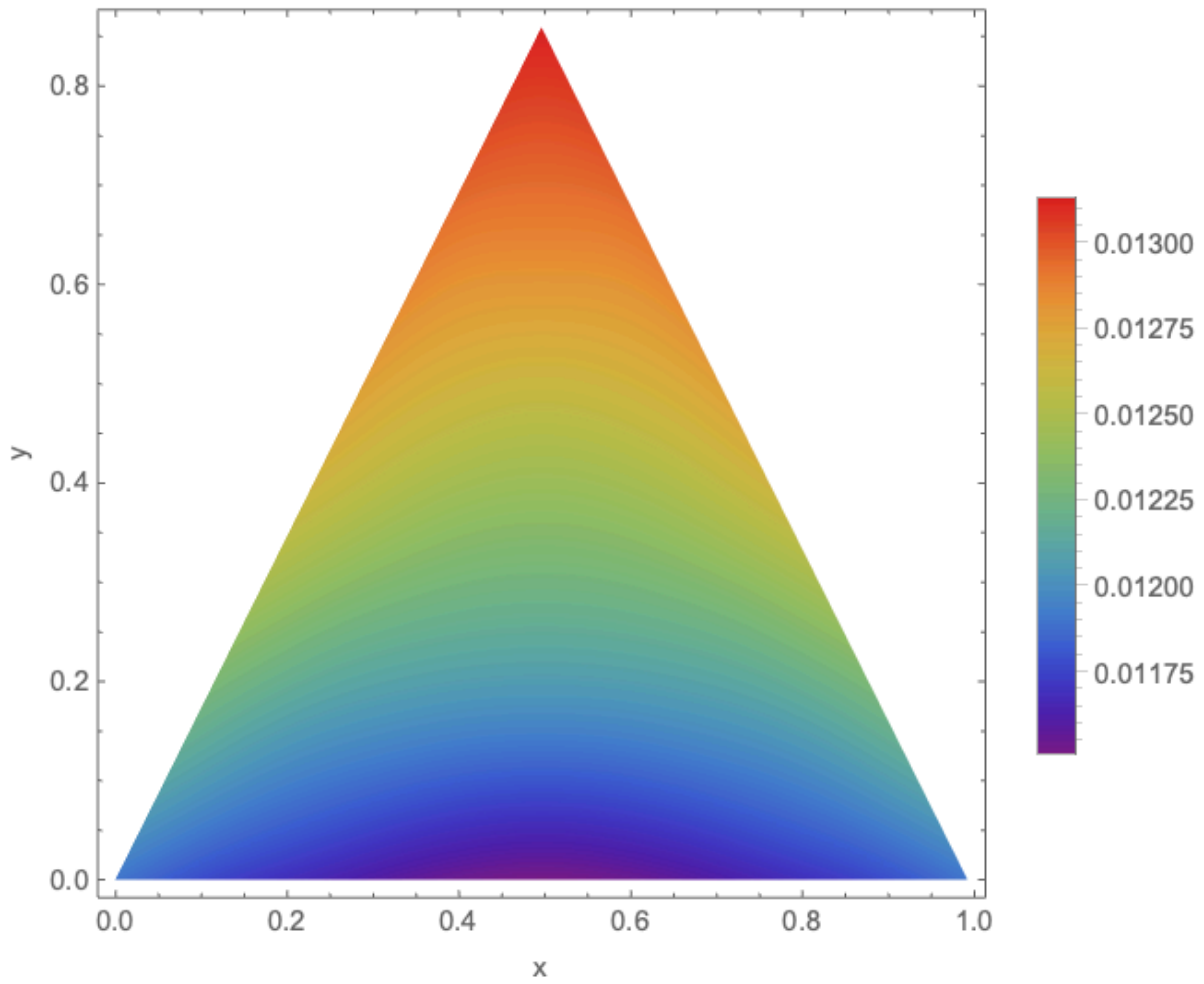
$$P_{ij} = \frac{1}{2}(1 + \hat e_{ij} \cdot \vec\sigma)$$

Smooth Link Weight K1 (K2 and K3 are rotated) before and after scalar smoothing

Projected From Icosahedron



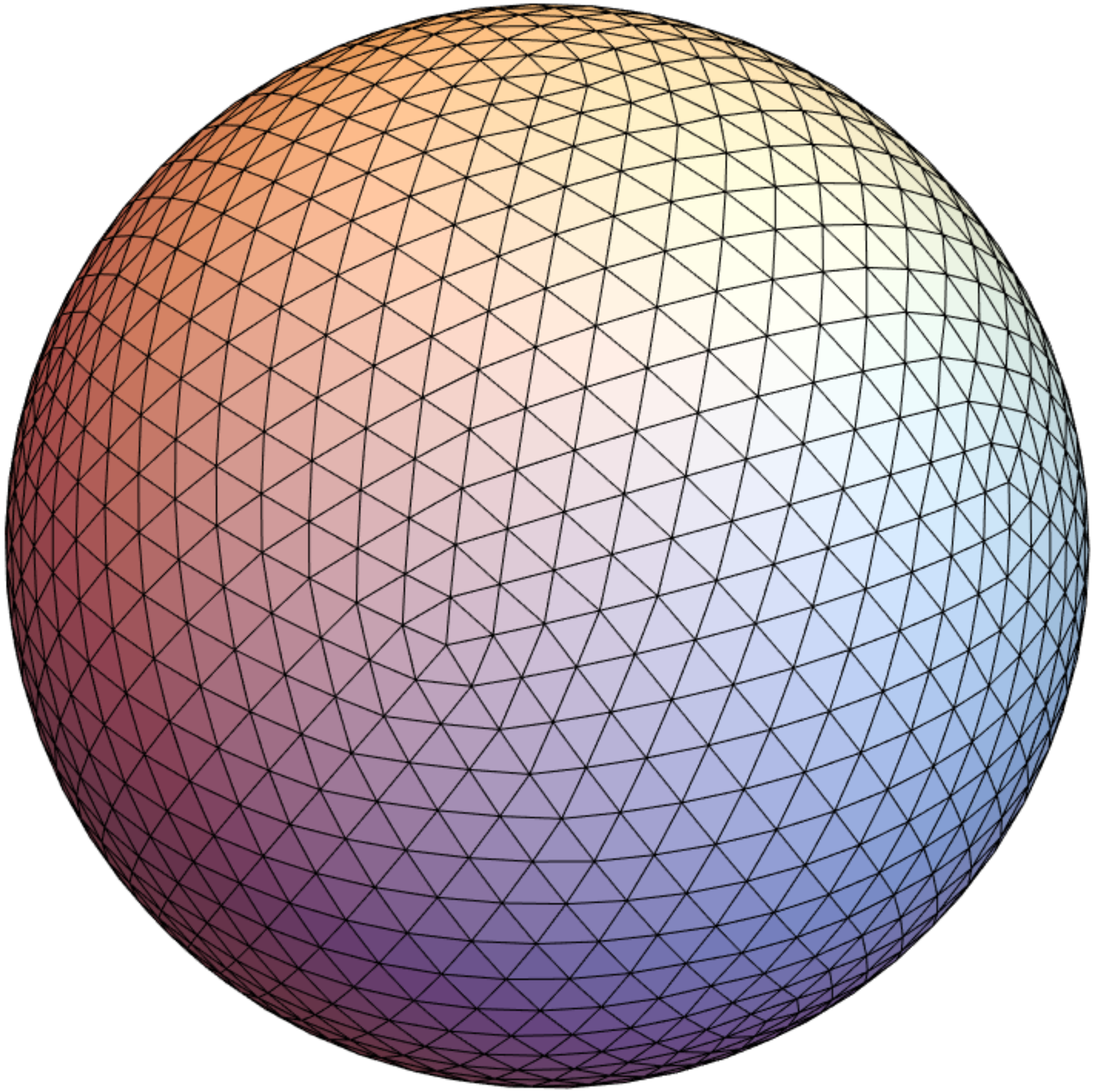
Smoothed Scalar Curvaturer



Back to Putting critical 2d Ising on the sphere

- Now set all the circumradii equal to converge to a differential affine tangent planes.

- Give we believe the Exact Ising CFT in the continuum limit.



SUMMARY OF SIMPLICIAL FIELDS

J = 0

$$S_{scalar} = \frac{1}{2}\sum_{\langle i,j\rangle} \frac{V_{ij}}{l_{ij}^2}(\phi_i - \phi_j)^2 \ ,$$

$$l_{ij}^2 = |\sigma_1(ij)|^2$$

J = 1/2

$$S_{Wilson} = \frac{1}{2}\sum_{\langle i,j\rangle} \frac{V_{ij}}{l_{ij}}(\bar{\psi}_i \hat{e}_a^{j(i)} \gamma^a \Omega_{ij} \psi_j - \bar{\psi}_j \Omega_{ji} \hat{e}_a^{i(j)} \gamma^a \psi_i)$$

J = 1

$$S_{gauge} = \frac{1}{2g^2 N_c}\sum_{\triangle_{ijk}} \frac{V_{ijk}}{A_{ijk}^2} Tr[2 - U_{\triangle_{ijk}} - U^\dagger_{\triangle_{ijk}}]$$

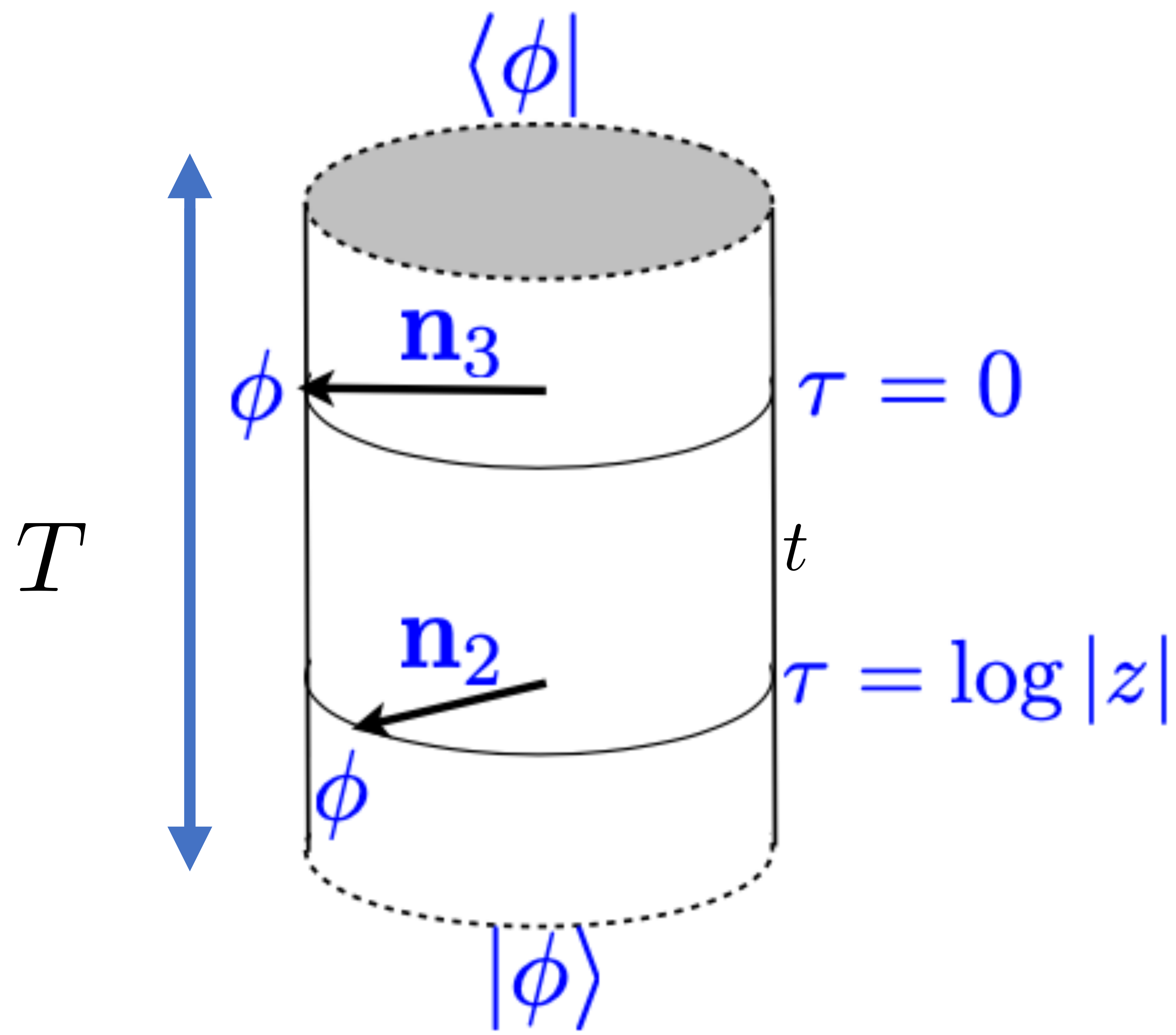
FFdual

$$\epsilon^{ijkl} Tr[U_{\triangle_{0ij}} U_{\triangle_{0kl}}] \simeq V_{ijkl}\epsilon^{\mu\nu\rho\sigma} Tr[F_{\mu\nu}(0) F_{\rho\sigma}(0)]$$

$$U_{\triangle_{ijk}} = U_{ij}U_{jk}U_{ki} \quad A_{ijk} = |\sigma_2(ijk)| \quad V_{ijk} = |\sigma_2(ijk) \wedge \sigma_2^*(ijk)|$$

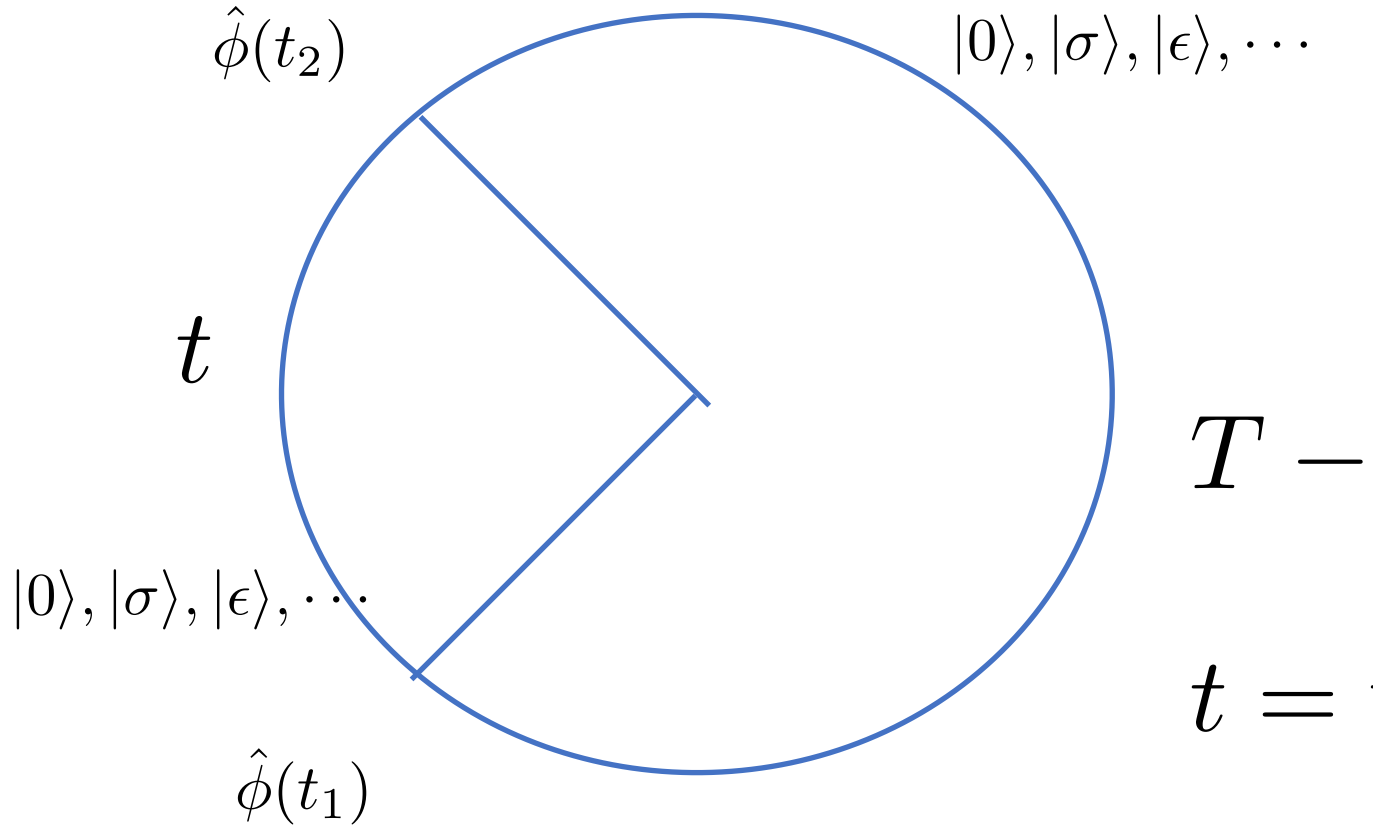
$$U_{0ij} = U_{0i}U_{ij}U_{j0} \quad , \quad U^\dagger_{0ij} = U_{0j}U_{ji}U_{i0} \quad V_{ij} = |\sigma_1(ij) \wedge \sigma_1^*(ij)|$$

Finite Volume/Temperature Measurements



$$Tr[e^{-\beta \hat{H} + \hat{h}_x \hat{\phi}_x}] = e^{-F(\beta, h_x)}$$

Central charge enters finite temperature free energy and amplitudes:
Can trace from UV to IR



$$t = t_2 - t_1$$

$$\langle \phi_\ell(t_2) \phi_{\ell_1}(t_1) \rangle_T = Tr[\, \hat{\phi}_\ell(0) e^{-t\hat{H}} \hat{\phi}_\ell(0) e^{-(T-t)\hat{H}}\,]$$
$$\equiv \sum_{\mathcal{O}} e^{-T\Delta_{\mathcal{O}}} \langle \mathcal{O}| \hat{\phi}_\ell(0) e^{-t(\hat{H} - \Delta_{\mathcal{O}})} \hat{\phi}_\ell(0) |\mathcal{O}\rangle$$
$$\simeq e^{-t\Delta_{\sigma,\ell}} + e^{-(T-t)\Delta_{\sigma,\ell}}$$
$$+ \quad f^2_{\epsilon\phi_\ell\sigma} e^{-\Delta_\sigma T} [e^{-t(\Delta_\epsilon - \Delta_\sigma)} + e^{-(T-t)(\Delta_\epsilon - \Delta_\sigma)}] + \cdots$$

$0 \le t \le T$ and periodic $t \to t + nT$

$\hat{H}|0\rangle = 0$ and $\hat{\phi}|0\rangle = |\sigma\rangle$