Exact lattice chiral symmetry

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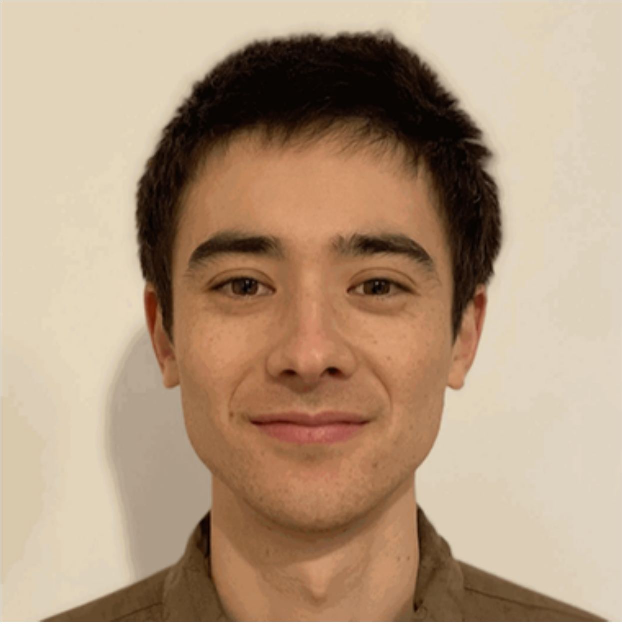
Based on arXiv:2370.17539 with



Evan Berkowitz @ IKP, Julich   Theo Jacobson @ UCLA

Massless fermions and their troubles

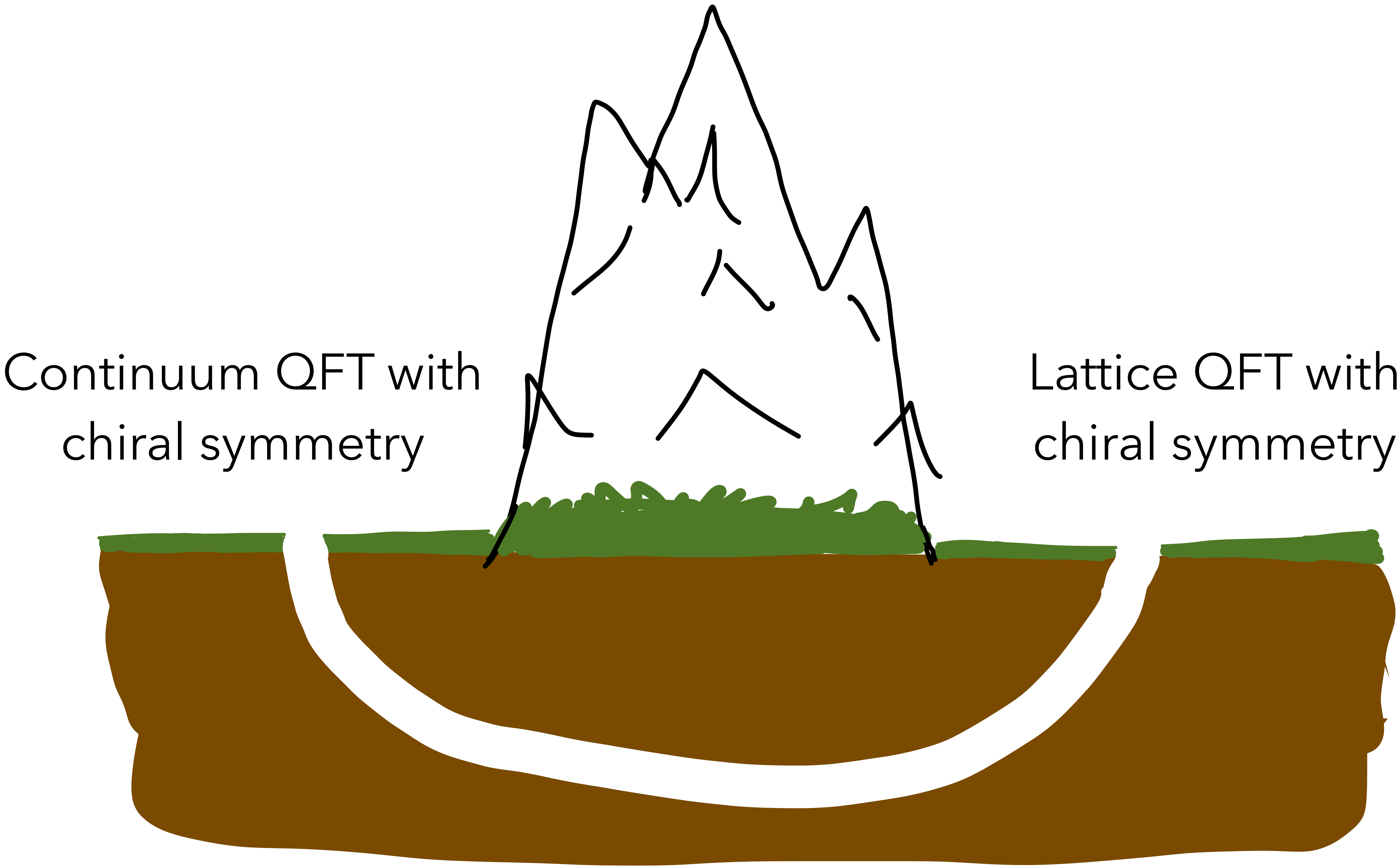
- **Challenge: discretize massless fermions while preserving their internal symmetries.**

  - Nielsen-Ninomiya theorem: can't discretize Dirac operator $\mathscr{D} = \gamma^\mu \partial_\mu$ while preserving

    1. Continuity in p_μ — Locality
    2. $\mathscr{D}(p) = \gamma^\mu p_\mu$ for $a|p| \ll 1$ — Free fermion as $a \to 0$
    3. $\mathscr{D}$ invertible except at $|p| \to 0$ — No doublers
    4. $\{\Gamma, \mathscr{D}\} = 0$ — Chiral symmetry

  - Wilson, staggered, and overlap/domain wall discretizations give up various combinations of these features.

Nielsen-Ninomiya Theorem



Bosonize, then discretize

The Idea

Berkowitz, AC, Jacobson, 2023

DeMarco, Lake, Wen, 2023

Morikawa, **Onoda**, Suzuki, (Friday)

- Discretizing $\gamma^\mu D_\mu$ is hard, but we actually **need** $\det(\gamma^\mu D_\mu)$:

$$Z = \int da_\mu d\psi d\bar{\psi} e^{-S[a_\mu, \psi, \bar{\psi}]}$$
$$= \int da_\mu \det(\gamma^\mu D_\mu) e^{-S[a_\mu]}$$

- There can be two expressions for $\det(\gamma^\mu D_\mu)$: one involving $\bar{\psi}, \psi$, and a `dual' expression from e.g. bosonization.
    - Discretizing the bosonized expression turns out to be easier than discretizing $\gamma^\mu D_\mu$!
- Gives lattice gauge theories with **exact** chiral symmetry.

The punchlines

- Today I'll explain **exact lattice chiral symmetry** in 2d $N_f = 1$ charge Q QED.
- Results generalize to many other U(1) gauge theory in 2d, including **chiral gauge theories**.
- Other generalizations:
  - Seems likely $d > 2$ is possible, work in progress!
  - Non-Abelian chiral symmetry much harder for now.

Chiral symmetry in charge Q 2d QED

$$\mathcal{L} = \frac{1}{4g^2} f_{\mu\nu}^2 + \bar{\psi}\left(\gamma^\mu \partial_\mu - iQa_\mu\right)\psi$$

- ABJ anomaly ⇒ axial current is not conserved.

$$\partial_\mu j_A^\mu = Q q_{\rm top}(a_\mu)\,,\ \ q_{\rm top}(a_\mu) = \frac{1}{2\pi}\epsilon^{\mu\nu} f_{\mu\nu}$$

- $q_{\rm top}(a_\mu) =$ topological charge density of the gauge field.

$$\Delta Q_A = 2Q \int d^2x\, q_{\rm top}(a_\mu) = 2Q\mathcal{I} \in 2Q\mathbb{Z}$$

- So $U(1)_A \to \mathbb{Z}_{2Q}$, but $\mathbb{Z}_2 \simeq (-1)^F \subset \mathbb{Z}_{2Q}$ is gauged, so

$$G_A = \frac{\mathbb{Z}_{2Q}}{\mathbb{Z}_2} \simeq \mathbb{Z}_Q$$

Bosonization of a 1+1d Dirac fermion

$\mathscr{L}_\psi = \bar{\psi}\gamma^\mu\partial_\mu\psi$ with gauged $(-1)^F$	=	Compact boson φ $\mathscr{L}_\varphi = \frac{1}{8\pi}(\partial_\mu\varphi)^2$

(In QED $(-1)^F$ is part of the U(1) gauge group)

- Evan Berkowitz (**next talk**!) will explain how to simulate the compact boson while preserving anomalies
- Gauging $U(1)_V$ should give an ABJ anomaly reducing $U(1)_A \to \mathbb{Z}_Q$.
- Challenge: seeing ABJ anomaly requires e.g. exact instanton number quantization, among other things.
    - Topological charges usually not manifest on lattice!

Compact boson review

$$\mathscr{L} = \frac{1}{8\pi}(d\varphi)^2, \ \varphi \simeq \varphi + 2\pi$$

- Axial current:

$$j_A = \frac{1}{4\pi} d\varphi = \frac{1}{4\pi} \partial_\mu \varphi dx^\mu$$

- Vector current:

$$j_V = \frac{-1}{2\pi} \star d\varphi = \frac{-1}{2\pi} \epsilon^{\mu\nu} \partial_\nu \varphi dx_\mu$$

- Axial charge:

$$Q_A(C) = \int_C \star j_A = \frac{1}{4\pi} \int_C \star d\varphi$$

φ shift charge

Conserved due to e.o.m.

- Vector charge:

$$Q_V(C) = \int_C \star j_V = \frac{1}{2\pi} \int_C d\varphi$$

φ winding charge

Conserved by Bianchi identity

Anomalies on the lattice

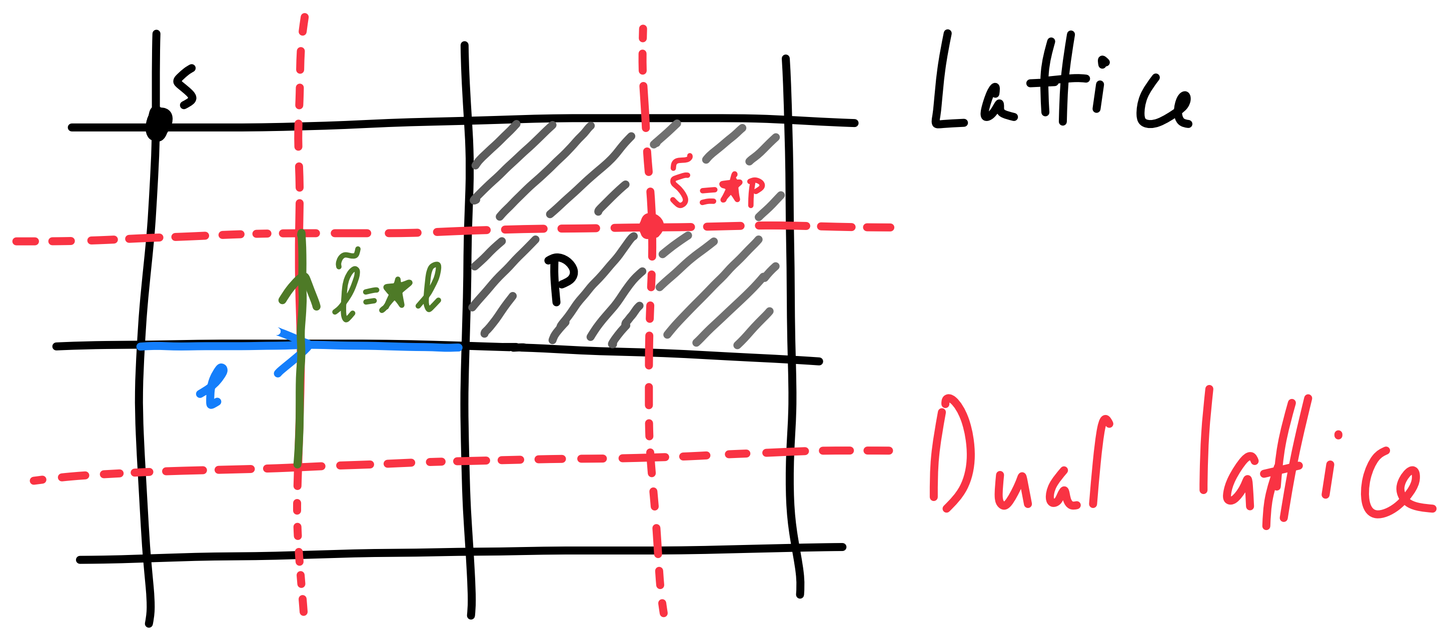
- Widespread historical view:
  - "Ultra-local lattice models can't capture anomalies."
- However, by now it is now well-known (by those who know it) that this isn't right:
  - Anomalies can be preserved on lattice, and even scalars can have anomalies.
  - Can preserve topological symmetries on lattice.

Sulejmanpasic, Gattringer; Shao, Seiberg, Gorantla; …
2019 - now

Cond-mat, hep-lat examples and antecedents:
Catterall et al (Friday), Singh+et al (Tuesday),
Lieb+Shutz+Mattis, Kitaev, Kapustin+Thorngren, …

The lattice

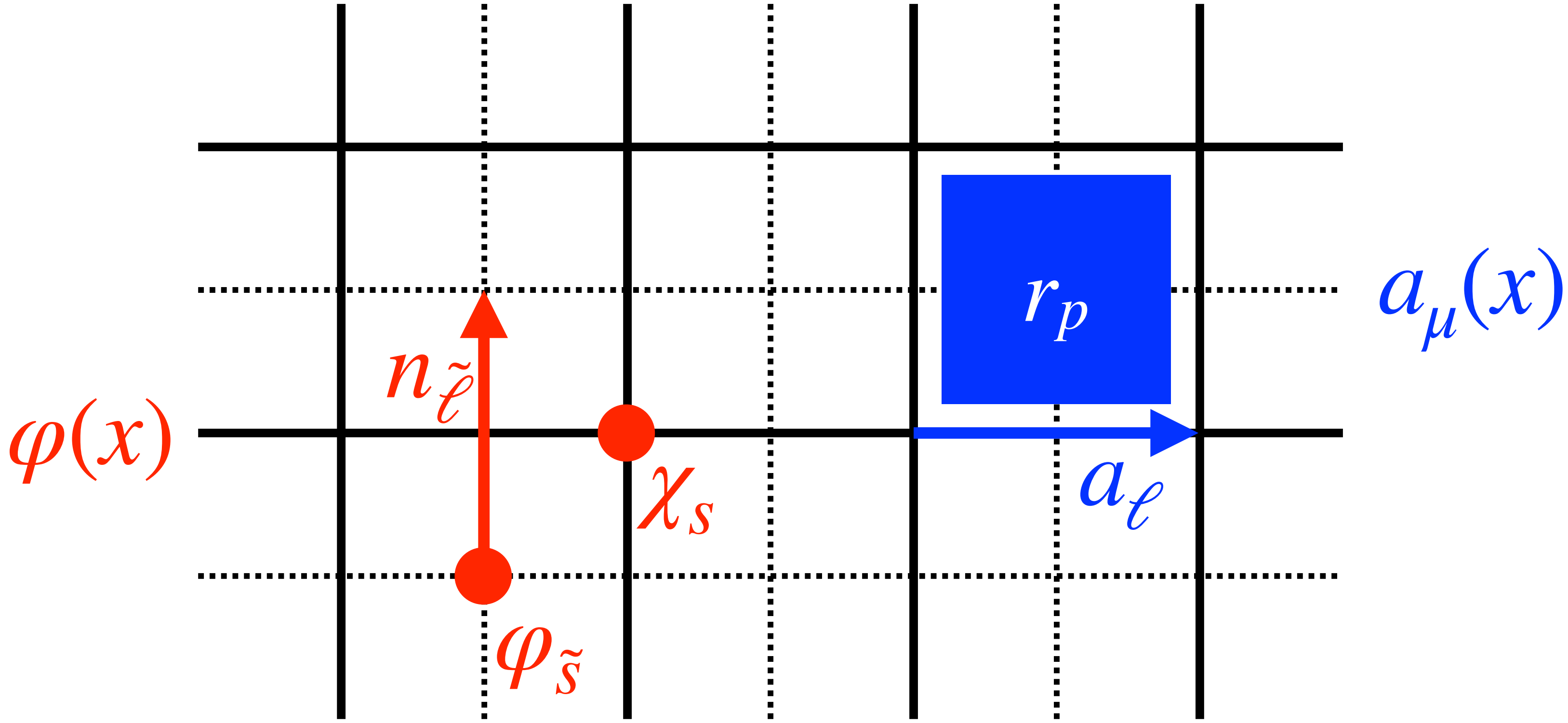
- Work on a square lattice with sites s, links ℓ, plaquettes p, and cells on dual lattice $\tilde{s}, \tilde{\ell}, \tilde{p}$.
- "Hodge star" map from lattice to dual lattice
  $\star\, s = \tilde{p}, \star\, \ell = \tilde{\ell}, \star\, p = \tilde{s}$
- $(d\omega)_{c^{r+1}} = \sum_{c^r \in \partial c^{r+1}} \omega_{c^r}$, so that $(d\phi)_\ell = \phi_{s+a\hat{\ell}} - \phi_s, \; d^2 = 0.$



Modified Villain formulation

Villain 1970s;
Gross, Klebanov 1990s;
Cheng, Sulejmanpasic,
Gattringer, Gorantla,
Fazzi, Lam, Seiberg,
Shao,…
2019 - now

- $U(1) \simeq \mathbb{R}/(2\pi\mathbb{Z})$, and it turns out a `more redundant' formulation with $\mathbb{Z}$ gauge field helps preserve global symmetries.



- Continuum $\varphi(x) \Rightarrow \{\varphi_{\tilde{s}} \in \mathbb{R}, \chi_s \in \mathbb{R}, n_{\tilde{\ell}} \in \mathbb{Z}\}$
- Continuum $a_\mu \Rightarrow \{a_\ell \in \mathbb{R}, r_p \in \mathbb{Z}\}$

Lattice action for 2d QED

$$S = \frac{\beta}{2}\left[(da)_p - 2\pi r_p\right]^2 + \frac{\kappa}{2}\left[(d\varphi)_{\tilde{\ell}} - 2\pi n_{\tilde{\ell}}\right]^2$$

$$- i\chi_s (dn)_{\star s} + \frac{iQ}{2\pi}\varphi_{\star p}\left[(da)_p - 2\pi r_p\right] - iQ a_\ell \, n_{\star \ell}$$

- χ_s integral sets $(dn)_{\tilde{p}} = (d \star j_V)_{\tilde{p}} = 0$. Q_V is conserved!
- Gauge transformations:

$$a_\ell \to a_\ell + (d\alpha)_\ell + 2\pi m_\ell$$

$$r_p \to r_p + (dm)_p$$

$$\varphi_{\tilde{s}} \to \varphi_{\tilde{s}} + 2\pi k_{\tilde{s}}$$

$$\chi_s \to \chi_s + 2\pi h_s + Q\alpha_s$$

$$n_{\tilde{\ell}} \to n_{\tilde{\ell}} + (dk)_\ell$$

$$\alpha_s \in \mathbb{R},\ m_\ell, k_{\tilde{s}}, h_{\tilde{s}} \text{ are all } \in \mathbb{Z}.$$

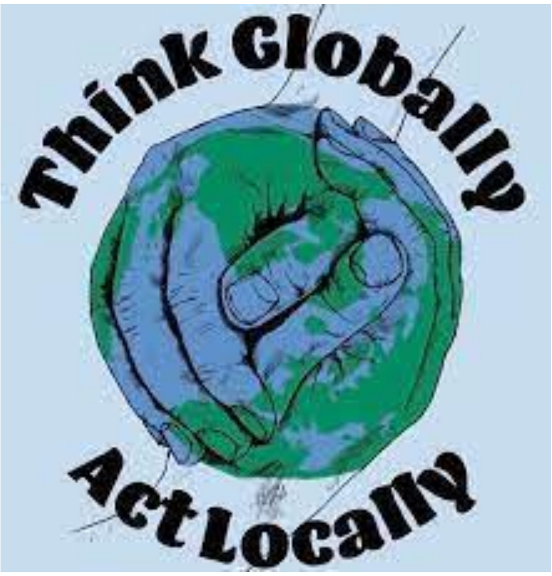
- Short calculation: despite appearance of "raw" a_ℓ, n_ℓ, χ_s fields in S, $\exp(-S)$ is gauge invariant!

Exact chiral symmetry

$$S = \frac{\beta}{2}\left[(da)_p - 2\pi r_p\right]^2 + \frac{\kappa}{2}\left[(d\varphi)_{\tilde{\ell}} - 2\pi n_{\tilde{\ell}}\right]^2$$
$$- i\chi_s (dn)_{\star s} + \frac{iQ}{2\pi}\varphi_{\star p}\left[(da)_p - 2\pi r_p\right] - iQ a_\ell\, n_{\star\ell}$$

- Quantization of Q_{top} on the lattice implies only $\varphi_{\tilde{s}} \to \varphi_{\tilde{s}} + \frac{2\pi k}{Q}, k = 1,\ldots, Q-1$ is a symmetry.

$$\Delta S = \sum_{p \in \text{spacetime}} \frac{iQ}{2\pi}\left[(da)_p - 2\pi r_p\right]\frac{2\pi k}{Q}$$
$$= -i2\pi k \sum_p r_p \in 2\pi i \mathbb{Z}$$



- Chiral symmetry acts **ultra-locally**, and ABJ anomaly is reproduced at finite lattice spacing.

Positives and negatives

- Negatives:
  - Only have results only in 1+1d - for now.
  - Only works for Abelian gauge theories - for now.
  - …
- Positives:
  - All symmetries act ultra-locally.
  - Capture anomalies on lattice for continuous and discrete symmetries, in gauge theories with massless fermions.
  - Discretization here is very different from usual ones.
  - Nielsen-Ninomiya theorem **does not** have an asterisk saying it doesn't apply to 1+1d Abelian models!
    - New evasion of the theorem seems interesting…

Is the construction practical?

- Some of you may be thinking I hid a really big \`negative': an apparently-horrible sign problem!

$$S_{\rm lat} = \frac{\kappa}{2}\left[(d\varphi)_{\tilde{\ell}} - 2\pi n_{\tilde{\ell}}\right]^2 + \frac{\beta}{2}\left[(da)_p - 2\pi r_p\right]^2$$
$$- i\chi_s (dn)_{\star s} + i\frac{Q}{2\pi}\left[(da)_p - 2\pi r_p\right]\varphi_{\star p}$$
$$- iQ a_\ell n_{\star \ell}$$

- Evan will say more about how to evade sign problems in this type of model.
- They are just mirages, disappearing on closer inspection.

drawception.com

Sign problem? What sign problem?

- Integrating over a_ℓ and r_p gives yet another dual action:

$$S_{\text{dual}} = \frac{\kappa}{2}\left[(d\varphi)_{\tilde{\ell}} - \frac{2\pi}{Q}(dt)_{\tilde{\ell}}\right]^2 + \frac{1}{2\beta}\left(\frac{Q}{2\pi}\right)^2\left(\varphi_{\tilde{s}} - \frac{2\pi}{Q}t_{\tilde{s}}\right)^2 - \frac{2\pi i}{Q}t_{\star p}(du)_p$$

- $t_s, u_\ell \in \mathbb{Z}$ appear during duality from S to S_{dual}, and integral over u_ℓ forces $(dt)_{\star\ell} = 0 \bmod Q$.
  - Can easily maintain this condition when doing Monte Carlo field updates. No sign problem in practice!
- S_{dual} contains precisely the expected `Schwinger boson'!

Chiral gauge theory

Berkowitz, AC, Jacobson, 2023

See also Onogi (Tuesday)

- Can discretize many 2d Abelian chiral gauge theories, with internal symmetries and their 't Hooft anomalies intact!
- Example: `3450' model, with left-handed Weyl fermions with charges 3,4 + right-handed Weyl fermions with charges 5,0.
  - There's an anomaly-preserving sign-problem-free bosonized formulation for 3450 model with gauged $(-1)^F$.
  - Surprise: lattice definition reveals an unexpected extra global symmetry at a special point in parameter space.
    - Symmetry is quite **exotic** in terms of fermions, not noticed before.
  - Excellent target for numerical exploration!

Conclusions

- Nielsen-Ninomiya theorem seems to kill hopes for **exact, ultra-locally-acting,** chiral symmetry on lattice.
- We've explored a new way around the Nielsen-Ninomiya theorem that accomplishes this goal.
  - It's numerically very cheap (see Evan's talk next!), and works for some chiral gauge theories.
  - Approach may generalize to $d > 2$, work in progress…
  - Many open questions!

Thanks for listening!